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Complete moment convergence for weighted sums of weakly dependent random variables and its application in nonparametric regression model☆

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ABSTRACT

In this paper, some results on the complete moment convergence for weighted sums of weakly dependent (or ρ^*-mixing) random variables are established. The results obtained in this paper improve and extend the corresponding one of Sung (2010). As an application of the main results, we present a result on complete consistency for the weighted estimator in a nonparametric regression model based on ρ^*-mixing errors.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and define the σ-algebras

$$\mathcal{F}_n^m = \sigma(X_k, n \leq k \leq m), \quad \text{and} \quad \mathcal{F}_S = \sigma(X_k, k \in S).$$

Firstly, let us recall the concept of ρ^*-mixing random variables.

Definition 1.1. A sequence $\{X_n, n \geq 1\}$ of random variables is called ρ^*-mixing ($\tilde{\rho}$-mixing, or weakly dependent) if for some integer $k \geq 1$, the mixing coefficient

$$\rho^*(k) = \sup_{S,T} \left(\sup_{X \in L^2(\mathcal{F}_S), Y \in L^2(\mathcal{F}_T)} \frac{|Cov(X,Y)|}{\sqrt{Var(X)Var(Y)}} \right) < 1,$$

where S and T are finite subsets of $\mathbb{N}$ such that $\text{dist}(S,T) \geq k$.

The concept of ρ^*-mixing random variables was introduced by Bradley (1992). ρ^*-mixing is similar to ρ-mixing, but they are quite different from each other in some ways. There are many articles for ρ^*-mixing sequences, we refer the readers to Utev and Peligrad (2003), and Shen et al. (2014) among others.

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The concept of complete convergence was first introduced by Hsu and Robbins (1947) as follows: a sequence $\{X_n, n \geq 1\}$ of random variables converges completely to a constant C if for all $\varepsilon > 0$, $\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty$. By the Borel–Cantelli lemma, the inequality above implies that $X_n \to C$ almost surely (a.s., for short).

Hsu and Robbins (1947) proved that the sequence of arithmetic means of independent identically distributed random variables converges completely to the expected value of the summands, provided the variance is finite. The converse was proved by Erdös (1949). This Hsu–Robbins–Erdös's result was generalized in different ways. Katz (1963), Baum and Katz (1965), and Chow (1973) obtained a generalization of complete convergence for a sequence of independent and identically distributed (i.i.d., for short) random variables with normalization of Marcinkiewicz–Zygmund type (see Gut, 1992). Chow (1988) first showed the complete moment convergence for a sequence of i.i.d. random variables by generalizing the result of Baum and Katz (1965). The concept of complete moment convergence is as follows:

Let $\{X_n, n \geq 1\}$ be random variables and $a_n > 0, b_n > 0, q > 0$. If $\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^q < \infty$ for all $\varepsilon > 0$, then X_n is said to be complete moment convergence. It is well known that complete moment convergence implies complete convergence. Thus, complete moment convergence is stronger than complete convergence. For more details about the complete moment convergence, we refer the readers to Wang and Hu (2014), Wu et al. (2014), Guo and Zhu (2013), Shen et al. (2016a) among others.

Recently, Sung (2010) obtained the following complete convergence for weighted sums of ρ^*-mixing random variables.

Theorem A. *Let $p > 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX = 0$. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n) \quad \text{for some } q > p. \tag{1.1}$$

If $E|X|^p < \infty$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_{ni} X_i \right| > \varepsilon n^{\alpha} \right) < \infty. \tag{1.2}$$

Conversely, if (1.2) holds for any array satisfying (1.1), then $E|X|^p < \infty$.

However, Sung (2010) did not consider the interesting case $\alpha p = 1$. The case $p > 1/\alpha$ and $\alpha > 1$ was also not considered. The main purpose of this paper is to improve and extend Theorem A from complete convergence to complete moment convergence for ρ^*-mixing random variables. In addition, the meaningful cases $\alpha p = 1$ and $p > 1/\alpha, \alpha > 1$ are also considered in this paper. It is deserved to mention that the method used to prove our main results is novel. As an application of our main results, we present a result on complete consistency for the weighted estimator in a nonparametric regression model based on ρ^*-mixing errors.

The layout of this paper is as follows. Some preliminary lemmas are provided in Section 2. Main results and their proofs are stated in Section 3. An application in nonparametric regression model of our main results is presented in Section 4. Throughout this paper, C represents some positive constant whose value may vary in different places. Let $\log x = \ln \max(x, e)$, and $I(A)$ be the indicator function of the set A. Denote $x^+ = xI(x \geq 0)$. $a \ll b$ implies that there exists some positive constant c_1 such that $a \leq c_1 b$. $\lfloor x \rfloor$ stands for the integer part of x. $a \vee b$ stands for $\max(a, b)$ and $a \wedge b$ means $\min(a, b)$.

2. Preliminary lemmas

To prove the main results of the paper, we need the following important lemmas. The first one comes from Sung (2009).

Lemma 2.1. *Let $\{a_i, 1 \leq i \leq n\}$ and $\{b_i, 1 \leq i \leq n\}$ be two sequences of real numbers. Then for any real number c, the following inequalities hold:*

$$(|a_i + b_i| - |c|)_+ \leq (|a_i| - |c|)_+ + |b_i|,$$
$$\left(\max_{1 \leq i \leq n} |a_i + b_i| - |c| \right)_+ \leq \left(\max_{1 \leq i \leq n} |a_i| - |c| \right)_+ + \max_{1 \leq i \leq n} |b_i|.$$

The following lemma is essential in proving our main results, which extends the corresponding one in Sung (2009) from $r \equiv 1$ to any $r > 0$.

Lemma 2.2. *Let $\{Y_i, 1 \leq i \leq n\}$ and $\{Z_i, 1 \leq i \leq n\}$ be sequences of random variables. Then for any $q > r > 0, \varepsilon > 0$, and $a > 0$, the following inequalities hold:*

$$E\left(\left| \sum_{i=1}^{n} (Y_i + Z_i) \right| - \varepsilon a \right)_+^r \leq C_r \left(\varepsilon^{-q} + \frac{r}{q-r} \right) a^{r-q} E \left| \sum_{i=1}^{n} Y_i \right|^q + C_r E \left| \sum_{i=1}^{n} Z_i \right|^r,$$