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Parisian quasi-stationary distributions for asymmetric Lévy processes

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ABSTRACT

In recent years there has been some focus on quasi-stationary behavior of an one-dimensional Lévy process X, where we ask for the law $\mathbb{P}(X_t \in dy|\tau_0^- > t)$ for $t \to \infty$ and $\tau_0^- = \inf\{t \geq 0 : X_t < 0\}$. In this paper we address the same question for so-called Parisian ruin time τ^θ, that happens when process stays below zero longer than independent exponential random variable with intensity θ.

1. Introduction

Let $X = \{X_t : t \geq 0\}$ be a spectrally one-sided Lévy process defined on the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ is assumed to satisfy the usual conditions for right continuity and completion. Suppose now that probabilities $\{\mathbb{P}_x\}_{x\in\mathbb{R}}$ correspond to the conditional version of $\mathbb{P}$ where $X_0 = x$ is given. We simply write $\mathbb{P}_0 = \mathbb{P}$.

Define the first passage time into the lower half line $(-\infty, 0)$ by

$$\tau_0^- = \inf\{t \geq 0 : X_t < 0\}.$$

In recent years there has been some focus on the existence and characterization of the so-called limiting quasi-stationary distribution (or Yaglom's limit) defined by:

$$\mu(dy) := \lim_{t\uparrow\infty} \mathbb{P}_x(X_t \in dy|\tau_0^- > t). \tag{1}$$

The sense in which this limit is quasi-stationary follows the classical interpretations of works such as Seneta and Vere-Jones (1966), Tweedie (1974), Iglehart (1974) (for a random walk), Jacka and Roberts (1995), Kyprianou (1971) (within the context

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of the $M/G/1$ queue), Martinez and San Martin (1994) (for a Brownian motion with drift), Kyprianou and Palmowski (2006) (for a general light-tailed Lévy process), Haas and Rivero (2012) (for regularly varying Lévy process), Mandjes et al. (2012) (for a workload process of single server queue) and other references therein.

In this paper, the principal object of interest is the quasi-stationary distribution of Parisian type, where τ_0^- is replaced by so-called Parisian ruin time:

$$\tau^\theta = \inf\{t > 0 : t - g_t > \mathbf{e}_\theta^{g_t}, X_t < 0\}, \tag{2}$$

where $g_t = \sup\{s \le t : X_s \ge 0\}$ and $\mathbf{e}_\theta^{g_t}$ is an independent of X exponential random variable with intensity $\theta > 0$ related to a separate negative excursion g_t. The ruin time τ^θ happens when process X_t stays negative longer than $\mathbf{e}_\theta^{g_t}$, which we will refer as implementation clock. We want to emphasize that in the definition of τ^θ there is not a single underlying random variable but a whole sequence of independent copies of a generic exponential random variable $\mathbf{e}_\theta^{g_t}$ each one of them attached to a separate excursion below zero. The model with exponentially distributed delay has also been studied by Landriault et al. (2014) and by Baurdoux et al. (2016). The name for this ruin comes from Parisian option that prices are activated or canceled depending on type of option if underlying asset stays above or below barrier long enough in a row (see Albrecher et al., 2012; Chesney et al., 1997; Dassios and Wu, 2010). So far only probability of Parisian ruin is known (see Dassios and Wu, unpublished manuscript; Czarna and Palmowski, 2011). In this paper we will find sufficient conditions for existence and identify the following limit:

$$\mu_x^\theta(dy) := \lim_{t\uparrow\infty} \mathbb{P}_x(X_t \in dy | \tau^\theta > t). \tag{3}$$

The idea of the proof of the main results is based on finding double Laplace transform of $\mathbb{P}_x(X_t \in dy, \tau^\theta > t)$ with respect to space and time. Then for some specific form of the Lévy measure (that will be defined later) using 'Heavyside' operation we will identify the asymptotics of this probability as $t \to \infty$ (see e.g. Abate and Whitt, 1997 and Henrici, 1977).

The paper is organized as follows. In Section 2 we state some preliminary facts. Later in Theorem 3 we give the formula, which is essential to obtain the main results of this paper. In Section 4 we present the quasi-stationary distribution for one-sided Lévy processes. Finally, in the Appendix the proof of Theorem 3 is presented.

2. Preliminaries

2.1. Asymmetric Lévy processes

In this section we present definitions, notations and basic facts on Lévy processes (we also refer to Bertoin, 1996 and Kyprianou, 2006 for a complete introduction to the theory of Lévy processes). We will focus on asymmetric Lévy processes, which are either spectrally positive (having nonnegative jumps) or spectrally negative (having nonpositive jumps). We denote by $\Pi_X(\cdot)$ the Lévy measure of X. With X we associate the Laplace exponent $\varphi(\beta) := \frac{1}{t}\log\mathbb{E}(e^{\beta X_t})$ defined for all β for which exists,

$$\varphi(\beta) = \gamma\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{-\beta z} - 1 + \beta z \mathbb{1}_{(-1,1)}(z)\right) \Pi_X(\mathrm{d}z),$$

for $\gamma \in \mathbb{R}$ and $\sigma \ge 0$. Moreover, we define function $\Phi(q) = \sup\{\beta \ge 0 : \varphi(\beta) = q\}$ called right-inverse of φ. We will also consider so-called dual process $\widehat{X}_t = -X_t$ with the Lévy measure $\Pi_{\widehat{X}}(0, y) = \Pi_X(-y, 0)$. Characteristics of $\widehat{X}$ will be indicated by using a hat over the existing notation for characteristics of X. In particular, the probabilities $\widehat{\mathbb{P}}_x$ and the expectations $\widehat{\mathbb{E}}_x$ concern the dual process.

For the process X we define the ascending ladder height process $(L^{-1}, H) = \{(L_t^{-1}, H_t)\}_{t\ge0}$:

$$L_t^{-1} := \begin{cases} \inf\{s > 0 : L_s > t\} & \text{if } t < L_\infty \\ \infty & \text{otherwise} \end{cases}$$

and

$$H_t := \begin{cases} X_{L_t^{-1}} & \text{if } t < L_\infty \\ \infty & \text{otherwise,} \end{cases}$$

where $L \equiv \{L_t\}_{t\ge0}$ is the local time at the maximum (see Kyprianou, 2006, p. 140). Recall that (L_t^{-1}, H_t) is a (killed) bivariate subordinator with the Laplace exponent $\kappa(\alpha, \beta) = -\frac{1}{t}\log\mathbb{E}\left(e^{-\alpha L_t^{-1} - \beta H_t}\mathbb{1}_{\{t\le L_\infty\}}\right)$ and with the jump measure Π_H. We define the descending ladder height process $(\widehat{L}^{-1}, \widehat{H}) = \{(\widehat{L}_t^{-1}, \widehat{H}_t)\}_{t\ge0}$ with the Laplace exponent $\widehat{\kappa}(\alpha, \beta)$ constructed from dual process $\widehat{X}$. Moreover, for a spectrally negative Lévy process from the Wiener–Hopf factorization we have:

$$\kappa(\alpha, \beta) = \Phi(\alpha) + \beta, \qquad \widehat{\kappa}(\alpha, \beta) = \frac{\alpha - \varphi(\beta)}{\Phi(\alpha) - \beta}; \tag{4}$$

see Kyprianou (2006, p. 169–170).