



On the concept of subcriticality and criticality and a ratio theorem for a branching process in a random environment

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ABSTRACT

We consider a branching process (Z_n) in a stationary and ergodic random environment $\xi = (\xi_n)$. Athreya and Karlin (1971) proved the basic result about the concept of subcriticality and criticality, by showing that under the quenched law $\mathbb{P}_\xi$, the conditional distribution of Z_n given the non-extinction at time n converges in law to a proper distribution on $\mathbb{N}^+ = \{1, 2, \dots\}$ in the subcritical case, and to the null distribution in the critical case, under the condition that the environment sequence is exchangeable. In this paper we first improve this basic result by removing the exchangeability condition on the environment, and by establishing a more general result about the conditional law of Z_n given the non-extinction at time $n + k$ for each fixed $k \geq 0$. As a by-product of the proof we also remove the exchangeability condition in another result of Athreya and Karlin (1971) for the subcritical case about the decay rate of the survival probability given the environment. We then establish a convergence theorem about the ratio $\mathbb{P}_\xi(Z_n = j)/\mathbb{P}_\xi(Z_n = 1)$, which can be applicable in each of the subcritical, critical, and supercritical cases.

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1. Introduction

A branching process in a random environment is a natural and important extension of the Galton–Watson process, where the offspring distributions vary from generation to generation according to a random environment. This model was first introduced by Smith and Wilkinson (1969) in 1969. Since then this model has attracted the attention of many authors, see for example the works by Athreya and Karlin (1971a,b), Afanasyev et al. (2005), Bansaye and Berestycki (2009), Böinghoff et al. (2010), Liang and Liu (2013), Huang and Liu (2014, 2012), Bansaye and Vatutin (2017), Grama et al. (2017), and the references therein.

For a Galton–Watson process (Z_n) , it is well-known (Athreya and Ney, 1972) that in the subcritical case, we have, for each $k \geq 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(Z_n = j | Z_{n+k} > 0) = b_{j,k}, \quad j = 1, 2, \dots \quad (1.1)$$

where $\{b_{j,k} : j \geq 1\}$ is a proper probability distribution on $\mathbb{N} = \{1, 2, \dots\}$, while in the critical case the limit above identically vanishes. This is the basic concept of subcriticality and criticality; it shows an essentially different limit behavior between the subcritical and critical cases, although in both cases the branching process becomes extinct almost surely.

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For $k = 0$, this result was extended by [Athreya and Karlin \(1971b\)](#) to a branching process in an exchangeable stationary and ergodic environment. As our first objective, in this paper we will not only improve the aforementioned result of [Athreya and Karlin \(1971b\)](#) for the case $k = 0$ by removing the exchangeability condition on the environment sequence, but also establish a general result valid for all $k \geq 0$, thus extending completely the corresponding result on the Galton–Watson process. Moreover, as a by-product, with the same technique we will also remove the exchangeability condition on the environment sequence of another result by [Athreya and Karlin \(1971b, Corollary 2\)](#) in the subcritical case, about the decay rate of the survival probability given the environment, and, more generally, for the decay rate of $1 - f_{\xi_0}(\cdots f_{\xi_n}(s) \cdots)$, where f_{ξ_n} is the progeny probability generating function at time n . The main idea is to remove the exchangeability condition by embedding the environment sequence ξ into a two-sided ergodic sequence, and then using the results of Athreya and Karlin for the time-reversed sequence $\eta_k = \xi_{-k}$, $k \geq 0$.

The second objective in this paper is to establish a ratio theorem for a branching process in a random environment. For a Galton–Watson process (Z_n) , the following interesting ratio theorem is known (see Section 1.7 in [Athreya and Ney, 1972](#)): for each $j \geq 1$, the limit

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}(Z_n = j)}{\mathbb{P}(Z_n = 1)} = \pi_j \quad (1.2)$$

exists, provided that $\mathbb{P}(Z_1 = 1) > 0$. This result is useful in the study of some other problems (see e.g. Theorem 1.8.1 in [Athreya and Ney, 1972](#)). In this paper we will extend this theorem to a branching process in a random environment.

The rest of the paper is organized as follows. In Section 2, we introduce our main results. [Theorems 2.1](#) and [2.3](#) are proved in Section 3, while [Corollary 2.2](#), [Theorem 2.4](#) are proved in Sections 4 and 5, respectively.

2. Main results

In this section we will present the main results of this paper.

Let us first describe the model. A branching process in a random environment can be defined as follows. Let $\xi = (\xi_0, \xi_1, \xi_2, \dots)$ be a stationary and ergodic sequence of random variables taking values in some space Θ , whose realization corresponds to a sequence of probability distributions on $\mathbb{N} = \{0, 1, 2, \dots\}$:

$$p(\xi_n) = \{p_i(\xi_n); i \geq 0\}, \quad \text{where } p_i(\xi_n) \geq 0 \quad \text{and} \quad \sum_{i=0}^{\infty} p_i(\xi_n) = 1.$$

A branching process $(Z_n)_{n \geq 0}$ in the random environment ξ is, by definition:

$$Z_0 = 1 \quad \text{and} \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \quad \text{for } n \geq 0,$$

where given the environment ξ , $X_{n,i}$ ($n \geq 0, i \geq 1$) is a sequence of (conditionally) independent random variables, each $X_{n,i}$ has distribution $p(\xi_n)$. Let

$$f_{\xi_n}(s) = \sum_{j=0}^{\infty} p_j(\xi_n) s^j, \quad s \in [0, 1],$$

be the probability generating function of $p(\xi_n)$, and

$$\phi_n(\xi; s) = f_{\xi_0}(f_{\xi_1}(\cdots f_{\xi_{n-1}}(s) \cdots)), \quad s \in [0, 1]. \quad (2.1)$$

Then $\mathbb{E}_{\xi}(s^{Z_n}) = \phi_n(\xi; s)$ is the conditional probability generating function of Z_n given the environment sequence ξ . Set for $n = 1, 2, \dots$,

$$\tilde{\phi}_n(\xi; s) = f_{\xi_{n-1}}(f_{\xi_{n-2}}(\cdots f_{\xi_0}(s) \cdots)), \quad s \in [0, 1]. \quad (2.2)$$

We now present the results of this paper.

Our first result concerns the basic notion of criticality and sub-criticality, already well discussed in the famous work by [Athreya and Karlin \(1971b\)](#): in their Theorems 2 and 4, they showed that in the subcritical case, the conditioned law of Z_n approaches a non degenerate limit law while for the critical case this limit is a null distribution. [Theorem 2.1](#) and [Corollary 2.1](#) improve the aforementioned results of [Athreya and Karlin \(1971b\)](#) by removing the exchangeability condition on the environment sequence.

Theorem 2.1. (i) If $\mathbb{E} \log f'_{\xi_0}(1) = 0$, then for each $s \in [0, 1)$, as $n \rightarrow \infty$,

$$\mathbb{E}_{\xi}(s^{Z_n} | Z_n > 0) \rightarrow 0 \quad \text{in law.} \quad (2.3)$$

(ii) If $\mathbb{E} \log f'_{\xi_0}(1) < 0$, then there exists a probability generating function $g(\xi; \cdot)$ such that, for each $s \in [0, 1)$, as $n \rightarrow \infty$,

$$\mathbb{E}_{\xi}(s^{Z_n} | Z_n > 0) \rightarrow g(\xi; s) \quad \text{in law.} \quad (2.4)$$

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