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2D general solution and fundamental solution for orthotropic thermoelastic materials

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ABSTRACT

The 2D general solution for the plane problem of thermoelastic materials is derived in terms of three harmonic functions using strict differential operator theory for the case of distinct eigenvalues. Based on the obtained general solution, the 2D fundamental solution for a steady line heat source in an infinite and a semi-infinite thermoelastic plane is obtained by three newly introduced harmonic functions. All components of coupled fields are expressed in terms of elementary functions and they are convenient to be used.

1. Introduction

Fundamental solutions or Green's functions play an important role in both applied and theoretical studies on the physics of solids. They are the foundations for lot of further works. For example, fundamental solutions can be used to construct many analytical solutions of practical problems when boundary conditions are imposed. They are essential in the boundary element method as well as the study of cracks, defects and inclusions. A great deal of work on this area can be found in the literature. When thermal effects are not considered, one can refer to the excellent works of Lifshitz and Rozentsveig [1], Elliott [2], Kroner [3], Willis [4], Sveklo [5], Lejcek [6], Pan and Chou [7], Banerjee and Butterfield [8].

When thermal effects are considered, Sharma [9] gave the fundamental solution of transversely isotropic thermoelastic materials in an integral form. Yu et al. [10] gave the Green's function for a point heat source in two-phase isotropic thermoelastic materials. Chen et al. [11] derived a compact 3D harmonic general solution for transversely isotropic thermoelastic materials. Based on this general solution, Hou et al. [12] obtained the 3D fundamental solution for a steady point heat source in an infinite orthotropic thermoelastic body. In addition, Hou et al. [13] obtained 2D Green's function for a steady line heat source in the interior of a semi-infinite orthotropic thermoelastic plane. However, the important 2D fundamental solution for a steady line heat source in an infinite orthotropic thermoelastic plane has not been presented so far.

As a further work, 2D fundamental solution for a steady line heat source in an infinite orthotropic thermoelastic plane is investigated in this paper. By the way, the solution for a steady line heat source on the surface of a semi-infinite orthotropic thermoelastic plane is also presented. For this object, the 2D general solution for the case of distinct eigenvalues, which is the most common case, is derived in Section 2. In Sections 3 and 4, three new suitable harmonic functions are constructed in the form of elementary functions with undetermined constants. The corresponding coupled field can be obtained by substituting these harmonic functions into the general solution. Finally, the paper concludes in Section 5.

2. 2D general solution for orthotropic thermoelastic material

If all components are independent of coordinate y, one will have the so-called plane problem in 2D Cartesian coordinates (x,z). When the principal material direction lies in x and z axes, the constitutive equations of orthotropic thermoelastic materials are in the form of

$$\sigma_x = c_{11}\frac{\partial u}{\partial x} + c_{13}\frac{\partial w}{\partial z} - \lambda_{11}\theta,$$
$$\sigma_z = c_{13}\frac{\partial u}{\partial x} + c_{33}\frac{\partial w}{\partial z} - \lambda_{33}\theta,$$
$$\tau_{zx} = c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{1}$$

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where u and w are components of the mechanical displacement in x and z directions, respectively; σ_{ij} are the components of stress; θ is the temperature increment; and c_{ij} and λ_{ii} are elastic constants and thermal modules, respectively.

In the absence of body forces, the mechanical and heat equilibrium equations are

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{zx}}{\partial z}=0, \quad \frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \sigma_z}{\partial z}=0, \tag{2a}$$

$$\left(\beta_{11}\frac{\partial^2}{\partial^2 x}+\beta_{33}\frac{\partial^2}{\partial^2 z}\right)\theta=0, \tag{2b}$$

where β_{ii} ($i=1,3$) are coefficients of heat conduction.

Substituting Eq. (1) into Eq. (2a) and combining the result with Eq. (2b), yields

$$\mathbf{D}[u \quad w \quad \theta]^T=0, \tag{3}$$

where $\mathbf{D}$ is a differential operator matrix defined by

$$\mathbf{D}=\begin{bmatrix} c_{11}\frac{\partial^2}{\partial x^2}+c_{44}\frac{\partial^2}{\partial z^2} & (c_{13}+c_{44})\frac{\partial^2}{\partial x\partial z} & -\lambda_{11}\frac{\partial}{\partial x} \\ (c_{13}+c_{44})\frac{\partial^2}{\partial x\partial z} & c_{44}\frac{\partial^2}{\partial x^2}+c_{33}\frac{\partial^2}{\partial z^2} & -\lambda_{33}\frac{\partial}{\partial z} \\ 0 & 0 & \beta_{11}\frac{\partial^2}{\partial^2 x}+\beta_{33}\frac{\partial^2}{\partial^2 z} \end{bmatrix}. \tag{4}$$

Eq. (3) is a homogeneous set of differential equations in u, w and θ. The general solution can be obtained routinely by the operator theory as

$$u=A_{i1}F, \quad w=A_{i2}F, \quad \theta=A_{i3}F \quad (i=1,2,3), \tag{5}$$

where A_{ij} ($i,j=1,2,3$) are the algebraic cominors of the matrix $\mathbf{D}$ of which the determinant is

$$|\mathbf{D}|=\left(a\frac{\partial^4}{\partial z^4}+b\frac{\partial^4}{\partial x^2\partial z^2}+c\frac{\partial^4}{\partial x^4}\right)\times\left(\beta_{11}\frac{\partial^2}{\partial^2 x}+\beta_{33}\frac{\partial^2}{\partial^2 z}\right), \tag{6}$$

where $a=c_{33}c_{44}$, $b=c_{11}c_{33}+c_{44}^2-(c_{13}+c_{44})^2$, $c=c_{11}c_{44}$. The function F in Eq. (5) should satisfy the following homogeneous equation:

$$|\mathbf{D}|F=0. \tag{7}$$

It can be seen that if $i=1$, 2 are taken in Eq. (5), two general solutions are obtained while $\theta=0$. These solutions are in fact identical to those without thermal effect and are not discussed here. Therefore, $i=3$ should be taken in Eq. (5) and the following general solutions are obtained:

$$u=\left(a_1\frac{\partial^2}{\partial x^2}+b_1\frac{\partial^2}{\partial z^2}\right)\frac{\partial F}{\partial x},$$
$$w=\left(a_2\frac{\partial^2}{\partial x^2}+b_2\frac{\partial^2}{\partial z^2}\right)\frac{\partial F}{\partial z},$$
$$\theta=\left(a\frac{\partial^4}{\partial z^4}+b\frac{\partial^4}{\partial x^2\partial z^2}+c\frac{\partial^4}{\partial x^4}\right)F, \tag{8}$$

where $a_1=-\lambda_{11}c_{44}$, $b_1=\lambda_{33}(c_{13}+c_{44})-\lambda_{11}c_{33}$, $a_2=\lambda_{33}c_{11}-\lambda_{11}(c_{13}+c_{44})$, $b_2=\lambda_{33}c_{44}$. F is the general solution of Eq. (7), which can be rewritten as

$$\prod_{j=1}^{3}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z_j^2}\right)F_j=0, \tag{9}$$

where $z_j=s_jz_j$ $s_3=\sqrt{\beta_{11}/\beta_{33}}$. s_j ($j=1,2$) are two roots (with positive real part) of the following algebraic equation:

$$as^4-bs^2+c=0. \tag{10}$$

As known from the generalized Almansi's theorem [14], the function F can be expressed in terms of three harmonic functions:

$$F=F_1+F_2+F_3 \quad \text{for distinct} \quad s_j \quad (j=1,2,3), \tag{11a}$$
$$F=F_1+F_2+zF_3 \quad \text{for} \quad s_1\neq s_2=s_3, \tag{11b}$$
$$F=F_1+zF_2+z^2F_3 \quad \text{for} \quad s_1=s_2=s_3, \tag{11c}$$

where F_j satisfy the following harmonic equation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z_j^2}\right)F_j=0 \quad (j=1,2,3). \tag{12}$$

The general solution for the case of distinct roots, which is the most common case, can be derived as follows:

$$u=\sum_{j=1}^{3}\kappa_{1j}\frac{\partial^3 F_j}{\partial x\partial z_j^2}, \quad w=\sum_{j=1}^{3}s_j\kappa_{2j}\frac{\partial^3 F_j}{\partial z_j^3}, \quad \theta=\kappa_{33}\frac{\partial^4 F_3}{\partial z_3^4}, \tag{13}$$

where

$$\kappa_{kj}=-a_k+b_ks_j^2 \quad (k=1,2), \quad \kappa_{33}=as_3^4-bs_3^2+c. \tag{14}$$

The general solution for other two cases can be derived in the same way.

Eq. (13) can be further simplified by letting

$$\kappa_{1j}\frac{\partial^2 F_j}{\partial z_j^2}=\psi_j. \tag{15}$$

Utilizing the above equation yields

$$u=\sum_{j=1}^{3}\frac{\partial\psi_j}{\partial x}, \quad w=\sum_{j=1}^{3}s_jk_{1j}\frac{\partial\psi_j}{\partial z_j}, \quad \theta=k_{23}\frac{\partial^2\psi_3}{\partial z_3^2}, \tag{16}$$

where

$$k_{1j}=\kappa_{2j}/\kappa_{1j}, \quad 5k_{23}=\kappa_{33}/\kappa_{13}. \tag{17}$$

The functions ψ_j still satisfy the following harmonic equation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z_j^2}\right)\psi_j=0 \quad (j=1,2,3). \tag{18}$$

Substituting general solution (16) into Eq. (1) yields

$$\sigma_x=\sum_{j=1}^{3}(-c_{11}+c_{13}s_j^2k_{1j}-\lambda_{11}k_{2j})\frac{\partial^2\psi_j}{\partial z_j^2},$$
$$\sigma_z=\sum_{j=1}^{3}(-c_{13}+c_{33}s_j^2k_{1j}-\lambda_{33}k_{2j})\frac{\partial^2\psi_j}{\partial z_j^2},$$
$$\tau_{zx}=\sum_{j=1}^{3}c_{44}(1+k_{1j})s_j\frac{\partial^2\psi_j}{\partial x\partial z_j}, \tag{19}$$

where $k_{21}=k_{22}=0$.

Substituting Eq. (19) into (2) by using Eq. (18), the following identities can be obtained:

$$c_{11}-c_{13}k_{1j}s_j^2+\lambda_{11}k_{2j}=c_{44}(1+k_{1j})s_j^2,$$
$$-c_{13}+c_{33}k_{1j}s_j^2-\lambda_{33}k_{2j}=c_{44}(1+k_{1j}),$$
$$(\beta_{11}-\beta_{33}s_j^2)k_{2j}=0 \quad (j=1,2,3). \tag{20}$$

By virtue of the above equations, the general solution (19) can be simplified as

$$\sigma_x=-\sum_{j=1}^{3}s_j^2\omega_j\frac{\partial^2\psi_j}{\partial z_j^2}, \quad \sigma_z=\sum_{j=1}^{3}\omega_j\frac{\partial^2\psi_j}{\partial z_j^2}, \quad \tau_{zx}=\sum_{j=1}^{3}s_j\omega_j\frac{\partial^2\psi_j}{\partial x\partial z_j}, \tag{21}$$

where

$$\omega_j=(c_{11}-c_{13}k_{1j}s_j^2+\lambda_{11}k_{2j})/s_j^2$$
$$=c_{44}(1+k_{1j})=-c_{13}+c_{33}k_{1j}s_j^2-\lambda_{33}k_{2j}. \tag{22}$$