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The Hadamard product and the free convolutions

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ABSTRACT

It is shown that if a probability measure ν is supported on a closed subset of $(0, \infty)$, that is, its support is bounded away from zero, then the free multiplicative convolution of ν and the semicircle law is absolutely continuous with respect to the Lebesgue measure. For the proof, a result concerning the Hadamard product of a deterministic matrix and a scaled Wigner matrix is proved and subsequently used. As a byproduct, a result, showing that the limiting spectral distribution of the Hadamard product is same as that of a symmetric random matrix with entries from a mean zero stationary Gaussian process, is obtained.

1. Introduction

In a recent paper (Chakrabarty and Hazra, 2016), it is shown that if ν is a probability measure such that $\nu([\alpha, \infty)) = 1$ for some $\alpha > 0$ and $\int_0^\infty x\nu(dx) < \infty$, then the free multiplicative convolution of ν and the semicircle law, defined below in (2.6), is absolutely continuous. In that paper, it is conjectured that the result should be true without the assumption that the mean is finite, although the methodology of that paper does not allow the removal of this assumption. This is the main goal of the current paper. Theorem 3.1 shows that if the probability measure ν is supported on a subset of the positive half line, which is bounded away from zero, then the free multiplicative convolution of ν and the semicircle law has a non-trivial semicircle component in the sense of free additive convolution. In other words, there exists a probability measure η such that

$$\nu \boxtimes \mu_1 = \eta \boxplus \mu_\alpha, \tag{1.1}$$

where μ_t is the semicircle law with standard deviation t, defined in (2.6) and $\boxtimes$ and $\boxplus$ denote the free multiplicative and additive convolutions respectively. This is precisely the result proved in Chakrabarty and Hazra (2016), albeit with the additional assumption that ν has finite mean. Theorem 3.1 and its corollary that $\nu \boxtimes \mu_1$ is absolutely continuous with respect to the Lebesgue measure, complement a corresponding result for the free additive convolution, proved in Biane (1997).

The proof of Theorem 3.1 is via the analysis of random matrices of the type

$$\left(f\left(\frac{i}{N+1}, \frac{j}{N+1}\right) X_{i\wedge j, i\vee j}/\sqrt{N} \right)_{1\le i,j\le N},$$

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where f is a function on $(0,1)^2$ satisfying certain regularity properties, and $\{X_{i,j} : 1 \le i \le j\}$ is a family of i.i.d. standard normal random variables. This random matrix is studied in Section 2, and the observations are summarized in Theorem 2.1. In Section 3, Theorem 2.1 is used to prove Theorem 3.1 which is the main result of this paper.

It turns out that the limiting spectral distribution obtained in Theorem 2.1 is same as that of a symmetric random matrix whose entries come from a stationary mean zero Gaussian process. Such random matrices were studied in Chakrabarty et al. (2016). The proof of this intriguing observation follows from equating the moments of the limiting spectral distributions obtained in the two models. This has been done in Section 4, and the observation mentioned above is stated as Theorem 4.1.

We conclude this section by pointing out the analogue of Theorem 3.1 in classical probability. The classical analogue is that if X and G are independent random variables, the latter following standard normal, then there exists a random variable Y independent of G such that

$$XG \stackrel{d}{=} \sigma Y + G,$$

if and only if

$$P(|X| \ge \sigma) = 1.$$

This can be proved using elementary probability tools. Theorem 3.1 is the free analogue of the *if* part of the above result. The author believes that the free analogue of the *only if* part is also true, that is, if (1.1) holds, then necessarily $\nu([\alpha,\infty)) = 1$. However, the methods of the current paper do not immediately prove the converse.

2. The Hadamard product

The following notations will be used throughout the paper. The (i,j)th entry of a matrix A will be denoted by $A(i,j)$. For two $m \times n$ matrices A and B, the Hadamard product of A and B, denoted by $A \circ B$, is defined as

$$(A \circ B)(i,j) := A(i,j)B(i,j), \quad 1 \le i \le m,\, 1 \le j \le n.$$

In other words, the Hadamard product is same as entry-wise multiplication.

Let $\mathcal{R}$ be the class of functions $f : (0,1) \times (0,1) \longrightarrow [0,\infty)$ such that

(1) for all $0 < \varepsilon < 1/2$, f is bounded on $[\varepsilon, 1-\varepsilon]^2$,
(2) the set of discontinuities of f in $(0,1)^2$ has Lebesgue measure zero,
(3) and $f(x,y) = f(y,x)$ for all $(x,y) \in (0,1)^2$.

Conditions (1) and (2) together are equivalent to assuming that f is Riemann integrable on any compact subset of $(0,1)^2$, and hence the letter 'R' has been used. However, $\mathcal{R}$ is strictly larger than the class of Riemann integrable functions on $(0,1)^2$ satisfying (3).

Fix $f \in \mathcal{R}$. For all $N \ge 1$, define a $N \times N$ matrix $A_{f,N}$ by

$$A_{f,N}(i,j) := f\left(\frac{i}{N+1}, \frac{j}{N+1}\right), \quad 1 \le i,j \le N. \tag{2.1}$$

Let $\{X_{i,j} : 1 \le i \le j\}$ be a family of i.i.d. standard normal random variables, and let W_N be a $N \times N$ scaled Wigner matrix formed by them. That is,

$$W_N(i,j) := N^{-1/2} X_{i\wedge j, i\vee j}, \quad 1 \le i,j \le N. \tag{2.2}$$

Define

$$Z_N := A_{f,N} \circ W_N, \quad N \ge 1. \tag{2.3}$$

The main result of this section is Theorem 2.1, which studies the LSD of Z_N, as $N \to \infty$. Before stating that result, we need to recall a few combinatorial notions. The reader can find a detailed discussion on these topics in Nica and Speicher (2006). For all $m \ge 1$, let $NC_2(2m)$ denote the set of all *non-crossing pair partitions* of $\{1,\ldots,2m\}$. Fix $m \ge 1$, and $\sigma \in NC_2(2m)$. Let $(V_1,\ldots,V_{m+1})$ denote the *Kreweras complement* of σ, that is the maximal partition $\overline{\sigma}$ of $\{\overline{1},\ldots,\overline{2m}\}$ such that $\sigma \cup \overline{\sigma}$ is a non-crossing partition of $\{1,\overline{1},\ldots,2m,\overline{2m}\}$. Note that the Kreweras complement of an element in $NC_2(2m)$ has exactly $(m+1)$ blocks, and hence is not a pair partition, although it is still non-crossing. For the sake of an unique labelling of the V_i's, we require that if $1 \le i < j \le m+1$, then the *maximal* element of V_i is smaller than that of V_j. Denote by $\mathcal{T}_\sigma$ the function from $\{1,\ldots,2m\}$ to $\{1,\ldots,m+1\}$ satisfying

$$i \in V_{\mathcal{T}_\sigma(i)}, \quad 1 \le i \le 2m.$$

The above notations have been introduced in Chakrabarty et al. (2016). For $\sigma \in NC_2(2m)$ and any function $f : (0,1)^2 \longrightarrow \mathbb{R}$, define a function $L_{\sigma,f} : (0,1)^{m+1} \longrightarrow \mathbb{R}$ by

$$L_{\sigma,f}(x_1,\ldots,x_{m+1}) := \prod_{(u,v)\in\sigma} f^2\left(x_{\mathcal{T}_\sigma(u)}, x_{\mathcal{T}_\sigma(v)}\right).$$