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Almost sure central limit theorem for self-normalized partial sums of ρ^{-}-mixing sequences

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ABSTRACT

Let $\{X, X_n\}_{n\in\mathbb{N}}$ be a weakly stationary sequence of ρ^{-}-mixing random variables. We discussed the almost sure central limit theorem for the self-normalized partial sums $S_n/\beta V_n$, where $S_n = \sum_{i=1}^{n} X_i$, $V_n^2 = \sum_{i=1}^{n} X_i^2$, constant $\beta > 0$. Our results generalize and improve those on almost sure central limit theorems obtained by previous authors from the independent case to ρ^{-}-mixing sequences and from $d_k = 1/k$ to $d_k = \ln \frac{c_k}{c_l} \exp(\ln^{\alpha} c_k)$, $0 \le \alpha < 1/2$.

1. Introduction

Definition 1.1 (*Joag-Dev and Proschan, 1983*). Random variables $X_1, X_2, \ldots, X_n$, $n \ge 2$ are said to be negatively associated (NA), if for every pair of disjoint subsets S and T of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}\{f(X_i, i \in S), g(X_j, j \in T)\} \le 0,$$

where $f, g \in \mathscr{C}$, $\mathscr{C}$ be a class of functions which are coordinatewise increasing and such that this covariance exists. A sequence of random variables $\{X_i\}_{i\in\mathbb{N}}$ is said to be NA if every finite subfamily is NA.

Definition 1.2 (*Kolmogorov and Rozanov, 1960*). Let $\{X_i\}_{i\in\mathbb{N}}$ is to be the random variables defined on a probability space $(\Omega, \mathscr{B}, \mathscr{P})$. Write $\mathscr{F}_S = \sigma(X_i, i \in S \subset \mathbb{N})$. Given σ-algebras $\mathscr{F}$, $\mathscr{R}$ in $\mathscr{B}$, Let

$$\rho(\mathscr{F}, \mathscr{R}) = \sup_{X\in L_2(\mathscr{F}), Y\in L_2(\mathscr{R})} \frac{|\mathbb{E}XY - \mathbb{E}X\mathbb{E}Y|}{\sqrt{\mathrm{Var}X\ \mathrm{Var}Y}}.$$

A sequence $\{X_i\}_{i\in\mathbb{N}}$ is said to be ρ^*-mixing, if $\rho^*(k) = \sup\{\rho(\mathscr{F}_S, \mathscr{F}_T) : S, T \subset \mathbb{N}, \mathrm{dist}(S, T) \ge k\} \to 0\ \ (k \to \infty)$.

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Definition 1.3 (*Zhang and Wang, 1999*). A sequence $\{X_i\}_{i\in\mathbb{N}}$ is to be ρ^--mixing, if $\rho^-(k) = \sup\{\rho^-(S,T) : S, T \subset \mathbb{N}, \mathrm{dist}(S,T) \geq k\} \to 0 \quad (k\to\infty)$, where

$$\rho^-(S,T) = 0 \vee \sup\left\{\frac{\mathrm{Cov}\left\{f(X_i, i\in S), g\left(X_j, j\in T\right)\right\}}{\sqrt{\mathrm{Var}\left\{f(X_i, i\in S)\right\}\mathrm{Var}\left\{g\left(X_j, j\in T\right)\right\}}}; f, g \in \mathscr{C}\right\}.$$

The concept of NA random variables introduced by Joag-Dev and Proschan (1983) and the definition of ρ^*-mixing random variables given by Kolmogorov and Rozanov (1960) have got more and more attention and a number of results have been gained. Zhang and Wang (1999) put forward the concept of ρ^--mixing random variables including NA and ρ^*-mixing random variables, which have a lot of applications. At recent time, many researchers started to study the properties of ρ^--mixing random variable sequences and got some results. For example, Zhang and Wang (1999) proved Rosenthal-type moment inequality and Marcinkiewicz–Zygmund law of large numbers, Zhang (2000a) gained the central limit theorems (CLT) of random fields, and Wang and Lu (2006) obtained the weak convergence theorems.

During the past two decades, several researchers focused on the almost sure central limit theorem (ASCLT) for partial sums S_n/σ_n of random variables which was started by Brosamler (1988) and Schatte (1988). The results from Brosamler (1988), Schatte (1988), Lacey and Philipp (1990), Ibragimov and Lifshits (1998), Berkes and Csáki (2001), Hörmann (2007) and Wu (2011b) were referenced. The classical limit theory usually contains unknown parameters, therefore, a universal way using statistics is to estimate the unknown parameters. If σ_n is replaced by an estimate from the given data, usually denoted by $V_n = \sqrt{\sum_{i=1}^n X_i^2}$. V_n is called a self-normalizer. A class of self-normalized has been proposed and studied in Peligrad and Shao (1995), Pena et al. (2009) and references therein. The past decade has witnessed a significant development on the limit theorems for the self-normalized partial sums S_n/V_n . After Huang and Pang (2010) proved ASCLT for self-normalized partial sums, the weight of ASCLT was improved by Wu (2012) as follows: Let $\{X, X_n\}_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables in the domain of attraction of the normal law with mean zero. Then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^n d_k I\left\{\frac{S_k}{V_k}\leq x\right\} = \Phi(x) \quad \text{a.s., for any } x\in \mathrm{R},$$

where $d_k = \frac{\exp(\ln^\alpha k)}{k}$, $D_n = \sum_{k=1}^n d_k$, $0 \leq \alpha < 1/2$ and $I(\cdot)$ denotes indicator function, $\Phi(\cdot)$ is the standard normal distribution function. We refer the reader to: Wu and Jiang (2016) and Wu (2017) for the almost sure central limit theorems for self-normalized version.

In purpose of this article is to study and establish the ASCLT, containing the general weight sequences, for self-normalized partial sums of ρ^--mixing sequence.

Throughout this paper, $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$, and c denotes a generic positive constant which may differ from one place to another.

Let $c_n > 0$ with

$$c_n \uparrow \infty, \quad \lim_{n\to\infty}\frac{c_{n+1}}{c_n} = 1 < \infty. \tag{1.1}$$

For every $1 \leq i \leq n$, Let

$$\bar{X}_{ni} := -\sqrt{n} I\left(X_i < -\sqrt{n}\right) + X_i I\left(|X_i| \leq \sqrt{n}\right) + \sqrt{n} I\left(X_i > \sqrt{n}\right),$$

$$\bar{S}_n := \sum_{i=1}^n \bar{X}_{ni}, \qquad \sigma_n^2 := \mathrm{Var}\bar{S}_n,$$

$$\bar{V}_n^2 := \sum_{i=1}^n \bar{X}_{ni}^2, \qquad \bar{V}_{n,1}^2 := \sum_{i=1}^n \bar{X}_{ni}^2 I\left(\bar{X}_{ni} \geq 0\right), \qquad \bar{V}_{n,2}^2 := \sum_{i=1}^n \bar{X}_{ni}^2 I\left(\bar{X}_{ni} < 0\right),$$

$$\delta_n^2 := \mathbb{E}\bar{X}_{n1}^2, \qquad \delta_{n,1}^2 := \mathbb{E}\bar{X}_{n1}^2 I\left(\bar{X}_{n1} \geq 0\right), \qquad \delta_{n,2}^2 := \mathbb{E}\bar{X}_{n1}^2 I\left(\bar{X}_{n1} < 0\right).$$

Apparently,

$$\mathbb{E}\bar{V}_n^2 = n\delta_n^2 = n\delta_{n,1}^2 + n\delta_{n,2}^2.$$

If $\{X_i\}_{i\in\mathbb{N}}$ is a sequence of ρ^--mixing, $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of functions which are coordinatewise increasing, then from the property 2 in Zhang and Wang (1999), $\{f_i(X_i)\}_{i\in\mathbb{N}}$ also is a sequence of ρ^--mixing. For each fixed n, because $\bar{X}_{ni}$ is monotonically increasing on X_i, $\left\{\bar{X}_{ni}\right\}_{1\leq i\leq n, n\geq 1}$ also is a sequence of ρ^--mixing. Further, $\left\{\bar{X}_{ni}^2 I\left(\bar{X}_{ni}\geq 0\right)\right\}_{1\leq i\leq n, n\geq 1}$ and $\left\{\bar{X}_{ni}^2 I\left(\bar{X}_{ni}< 0\right)\right\}_{1\leq i\leq n, n\geq 1}$ also is a sequence of ρ^--mixing. Our main theorem is as follows.