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Strong existence and uniqueness to a class of nonlinear SPDEs driven by Gaussian colored noises

Jie Xiong [a], Xu Yang [b,*]

[a] *Department of Mathematics, Faculty of Sci & Tech., University of Macau, Macau, China*
[b] *School of Mathematics and Information Science, North Minzu University, Yinchuan, China*

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ABSTRACT

In this paper, we establish the strong existence and uniqueness to the following stochastic partial differential equation:

$$u_t(x) = u_0(x) + \frac{1}{2}\int_0^t \Delta u_s(x)\mathrm{d}s + \int_0^t \int_{\mathbb{R}} g(u_s(x), \nabla u_s(x))\mathrm{d}F(s, y),$$

where $F(s, x)$ is a Gaussian noise on $\mathbb{R}_+ \times \mathbb{R}$ that is white in time and colored in space. This gives an affirmative answer to the open problem in Gomez et al. (2013) for certain non-Lipschitz condition of g.

1. Introduction and main result

For $k \geq 1$ let $\mathcal{X}_k$ be the Hilbert space consisting of all functions f so that $f^{(k)} \in L^2(\mathbb{R}, \mathrm{e}^{-|x|}\mathrm{d}x)$, where $f^{(k)}$ is the kth order derivative in the sense of generalized functions. In this paper we always assume that all random elements are defined on a filtered complete probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual hypotheses. Our aim of this paper is to study the strong existence and uniqueness for the following $\mathcal{X}_1$-valued stochastic partial differential equation (SPDE):

$$u_t(x) = u_0(x) + \frac{1}{2}\int_0^t \Delta u_s(x)\mathrm{d}s + \int_0^t \int_{\mathbb{R}} g(u_s(x), \nabla u_s(x))\mathrm{d}F(s, y), \tag{1.1}$$

where ∇ and Δ denote the first and the second order spatial differential operators, respectively, $F(s, x)$ is a mean-zero Gaussian colored noise with the covariance function

$$\mathbf{E}\{F(t, x)F(s, y)\} = \delta(t - s)\sigma(x, y)$$

and σ is a nonnegative bounded function on $\mathbb{R} \times \mathbb{R}$.

The strong uniqueness to (1.1) was proved in Gomez et al. (2013) by transforming the SPDE into a backward doubly stochastic differential equation as $g(x, y)$ satisfies certain non-Lipschitz condition in x and Lipschitz condition in y. But the existence was left as an open problem. In this paper we give an affirmative answer to this problem for a certain case.

If $g(x, y)$ is independent of y, Mytnik et al. (2006) showed that (1.1) has a strong unique solution as g is Hölder continuous for certain range of index. For the case $g(x, y) \equiv |x|^{\gamma}$ and the Gaussian white noise F, Mytnik and Perkins (2011) proved that

* Corresponding author.
E-mail addresses: jiexiong@umac.mo (J. Xiong), xuyang@mail.bnu.edu.cn (X. Yang).

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(1.1) has a strong unique solution as $\gamma > 3/4$, Mueller et al. (2014) showed a non-uniqueness result as $1/2 \le \gamma < 3/4$ and Burdzy et al. (2010) gave a non-uniqueness result as $0 < \gamma < 1/2$. If $g(x, y) \equiv (|x|^\beta \wedge 1)y$, then the main result of this paper implies that (1.1) has a strong unique solution for all $\beta \ge 0$ (see Example 1.2 in the following), which is interesting.

Eq. (1.1) is a form SPDE, it can be understood in the following sense: for each $f \in \mathscr{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}$,

$$\int_{\mathbb{R}} u_t(x)f(x)\mathrm{d}x = \int_{\mathbb{R}} u_0(x)f(x)\mathrm{d}x + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}} u_s(x)f''(x)\mathrm{d}x + \int_0^t \int_{\mathbb{R}} g(u_s(x), \nabla u_s(x))f(x)\mathrm{d}F(s, x).$$

Due to technical reason, we will obtain the existence of the solution under a condition which is different from that imposed in Gomez et al. (2013) for the uniqueness of the solution. Keeping in mind the motivating example $(|x|^\beta \wedge 1)y$, we assume throughout this article that

$$g(x, y) = \overline{g}(x, y) + \widetilde{g}(x)y.$$

For the convenience of the statements of the result, let us formulate the following conditions on $\overline{g}$ and $\widetilde{g}$:

(a) (Lipschitz condition) There are constants $K_1 > 0$ and $\alpha > 0$ so that

$$|\overline{g}(x_1, y_1) - \overline{g}(x_2, y_2)| \le K_1|x_1 - x_2| + \alpha|y_1 - y_2|, \quad x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

(b) There is a constant $K_2 > 0$ so that $|\widetilde{g}(x)| \le K_2$ for all $x \in \mathbb{R}$.

Theorem 1.1. *Suppose that conditions (a,b) hold and $\|\sigma\|_{2,\infty} < \infty$ with*

$$\|\sigma\|_{k,\infty} := \sum_{i,j\ge 0,\ i+j\le k} \sup_{x,y} |\partial^{i+j}_{x^i y^j}\sigma(x, y)|, \quad k \ge 0.$$

If $u_0 \in \mathcal{X}_1$ and $2\alpha^2\|\sigma\|_{2,\infty} < 1$, then (1.1) has a strong unique $\mathcal{X}_1$-valued solution.

Example 1.2. *If $g(x, y) = (|x|^\beta \wedge 1)y$ for $\beta \ge 0$ and $\|\sigma\|_{2,\infty} < \infty$, SPDE (1.1) has a strong unique solution. Note that the uniqueness was obtained in Gomez et al. (2013) for $\beta \in \left[\frac{1}{2}, 1\right]$ only.*

2. Proof of Theorem 1.1

We define the Hilbert norm $\|\cdot\|$ for $\mathcal{X}_0$ by

$$\|f\| = \int_{\mathbb{R}} |f(x)|^2 e^{-|x|}\mathrm{d}x.$$

We denote the corresponding inner product by $\langle\cdot,\cdot\rangle$. Define $J(x) = \int_{\mathbb{R}} e^{-|y|}\rho_0(x-y)dy$ with the mollifier ρ_0 given by

$$\rho_0(x) = C' \exp\big(-1/(1-x^2)\big)1_{\{|x|<1\}},$$

where C' is a constant so that $\int_{\mathbb{R}} \rho_0(x)dx = 1$. By (2.1) in Mitoma (1985), for each $n \ge 0$ there are constants $C_n', C_n'' > 0$ so that

$$C_n'' e^{-|x|} \le |J^{(n)}(x)| \le C_n' e^{-|x|}, \quad x \in \mathbb{R}, \tag{2.1}$$

which implies

$$|J^{(n)}(x)| \le C_n J(x), \quad x \in \mathbb{R} \tag{2.2}$$

for some constant $C_n > 0$. We may and will replace $e^{-|x|}$ by $J(x)$ in the definitions of space $\mathcal{X}_k$ and the norm $\|\cdot\|$. In this section we always use C and $C(\varepsilon)$ to denote positive constants whose values might change from line to line and independent of the constant n and s. Let $(P_t)_{t\ge 0}$ denote the transition semigroup of a one-dimensional Brownian motion. For $t > 0$ and $x \in \mathbb{R}$ write $p_t(x) := (2\pi t)^{-\frac{1}{2}} \exp\{-x^2/(2t)\}$. We will need the following lemma.

Lemma 2.1.

(i) *Let*

$$K_3 = \left|\int_{\mathbb{R}} e^z p_1(z)\mathrm{d}z\right|^{\frac{1}{2}}.$$

Then $\|P_t f\| \le K_3^t\|f\|$ for each $f \in \mathcal{X}_0$ and $t \in [0, 1]$. Moreover, there is a constant $K_4 > 0$ so that

$$\|P_t f - P_s f\| \le K_4\|f'\|\sqrt{|t-s|}$$

for $t, s \in [0, 1]$ and $f \in \mathcal{X}_1$.