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A generalization of Gerber's inequality for ruin probabilities in risk-switching models

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ABSTRACT

In this paper, we investigate a risk-switching Sparre Andersen model which generalizes several discrete time- as well as continuous time risk models. A Markov chain is used as a 'switch' under the assumption that jumps change the amount and/or respective wait time distributions of claims while the insurer can adapt the premiums in response. A generalized Gerber-type inequality for the vector of ruin probabilities is proven showing that the risk-switching models allow sophisticated mathematical results in spite of their complexity.

1. Introduction

All stochastic objects considered in the paper are assumed to be defined on a probability space $(\Omega; \mathcal{F}; \mathbb{P})$. Using a martingale approach, Gerber (1979) proved a well-known upper bound for the ruin probabilities in finite horizon. In this paper, we improve this bound as well as generalize it to the risk-switching Sparre Andersen model. To be more precise, let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{R}$ – the real line. Set $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}^1 = \mathbb{N} \backslash \{1\}$, $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}^0_+ = [0, \infty)$. Let a random variable X_k denote the amount of the kth claim, T_1 – the moment when the first claim appears and T_k – the time between the $(k-1)$th claim and the kth one, $k \in \mathbb{N}^1$. Let us denote by A_n the moment when the nth claim appears. With this notation, $A_n = T_1 + \cdots + T_n,\ n \in \mathbb{N}^0$, under the convention that $A_0 = 0$. Let a random variable C_k denote the insurer's premium rate at the time interval $[A_{k-1}, A_k)$. The random variables C_k, T_k and X_k are assumed to be positive (a.s.) for each $k \in \mathbb{N}$. Throughout this paper, we will assume that the distributions of C_k, T_k and X_k have no singular parts.

Let $\{I_k\}_{k \in \mathbb{N}^0}$ be a Markov chain with

1. a finite state space $S = \{1, 2, \ldots, s\},\ s \in \mathbb{N}$;
2. an initial distribution $(p_i)_{i \in S}$ such that the probabilities $p_i = \mathbb{P}(I_0 = i)$ are positive for each $i \in S$;
3. a transition matrix $P = \left(p_{ij}\right)_{i,j \in S}$ such that the probabilities $p_{ij} = \mathbb{P}\left(I_{k+1} = j | I_k = i\right)$ are positive for all $i, j \in S$.

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The probabilities p_{ij} are assumed to be independent of $k \in \mathbb{N}^0$. The jumps are allowed to appear at the moments $\{A_k\}_{k\in\mathbb{N}}$ only. Each jump changes the distribution of T_k and/or X_k if $i \neq j$, so we will interpret $\{I_k\}_{k\in\mathbb{N}^0}$ as a 'switch'. We assume that the insurance premium rate $C_k = c(I_{k-1})$, where c is a known positive function defined on S.

The continuous-time risk theory in a Markovian environment is quite well-known, see, e.g. Asmussen and Albrecher (2010). This approach was earlier used in Reinhard (1984) and Asmussen (1989), where the detailed references to queuing theory can be found. Since then, several regime-switching models have been studied (see e.g. Wang et al., 2016; Landriault et al., 2015; Chen et al., 2014 and Guillou et al., 2013).

The main difference between these models and the one investigated in the present paper lies in the moment of a 'switch'. In the above model, switches are allowed to appear only at the moments when successive claims arrive. This assumption causes no difficulties from the perspective of large insurance companies where claims appear almost incessantly. However, from the theoretical point of view, it has an essential advantage. Namely, it enables to treat in a unified way several discrete- and continuous time models leading to a generalized Gerber-type inequality for ruin probabilities. Special cases of particular interest are as follows: a discrete time risk-switching model, a risk-switching model with exponentially distributed wait times, the Sparre Andersen model, the Cramér–Lundberg model and a discrete time risk model without a switch.

Set $Z_k = X_k - c(I_{k-1})T_k$, $k \in \mathbb{N}$. Let a non-negative real u denote the insurer's surplus at 0 and $U_n = U(n, u)$ — at the moment A_n, respectively. The *surplus process* (*risk process*) $\{U_n\}_{n\in\mathbb{N}}$ is given by

$$U_n = u - \sum_{k=1}^{n} Z_k. \tag{1}$$

The *time of ruin*

$$\tau = \tau(u) = \inf\{n \in \mathbb{N} : U(n, u) < 0\} \tag{2}$$

is the first time when the insurer's surplus falls below zero (here $\inf \emptyset$ means ∞). The conditional probability that $\tau(u)$ is not greater than n, given the initial state i, considered as a function of the initial surplus u, is called the probability of ruin at or before the nth claim. We will denote it by $\Psi_n^i(u)$, under the convention that

$$\Psi_0^i(u) = 0, \quad i \in S. \tag{3}$$

In contrast to risk models without a switch, one has to consider several ruin probabilities $\Psi_n^1, \ldots, \Psi_n^s$ and a vector formed by them. Set

$$\boldsymbol{\Psi}_n(u) = (\Psi_n^1(u), \ldots, \Psi_n^s(u)), \tag{4}$$

where $n \in \mathbb{N}^0$ and $u \geqslant 0$.

The conditional distribution of X_1 (respectively, T_1), given the initial state i and the state j at the moment A_1, will be denoted by F^{ij} (respectively, G^{ij}). Recall that the premium rate C_1, given the initial state i, equals a positive real $c(i)$. Let

$$M^i(r) = \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_0^\infty e^{-r(c(i)t-x)} dF^{ij}(x) dG^{ij}(t), \quad i \in S,\ r \in \mathbb{R}. \tag{5}$$

Positive constants $r_0^1, \ldots, r_0^s$ are assumed to satisfy the following equations:

$$M^i(r_0^i) = 1, \quad i \in S, \tag{6}$$

forming a vector-type counterpart of the so-called *adjustment coefficient*. It will be called an *adjustment vector*.

In Section 3, we will prove the following inequality:

$$\Psi_n^i(u) \leqslant \inf_{r \geqslant r_0^*} \{e^{-ru} M^i(r)[M^*(r)]^{n-1}\}, \quad i \in S,\ n \in \mathbb{N},\ u \geqslant 0, \tag{7}$$

where $r_0^* = \min_{i\in S}\{r_0^i\}$ and $M^*(r) = \max_{i\in S}\{M^i(r)\}$ for $r \geqslant r_0^*$.

The well-known Gerber's inequality (see (17)) follows from a one-dimensional version of (7). Gerber's inequality can find applications, for instance, in optimizing reinsurance contracts, see, e.g. Centeno (2002).

Other risk-switching models applied to finance and economics can be found e.g. in Frühwirth-Schnatter (2006). Markov additive processes are studied in Asmussen (2003) or Feng and Shimizu (2014) among others. The Markov-modulated Poisson process can be found, for instance, in Reinhard (1984) and Asmussen (1989), Asmussen and Albrecher (2010) or Guillou et al. (2013). For a comprehensive treatment of extreme values methodology for many standard insurance models, we refer the reader to Embrechts et al. (2001). Taylor (1976) was, to the best of our knowledge, the first researcher who used the operator approach to evaluate ruin probabilities. Since then, this method has been further developed and generalized by Gajek (2005) and Gajek and Rudź (2013).