Statistics and Probability Letters 129 (2017) 260–268

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Asymptotic results for a multivariate version of the alternative fractional Poisson process ☆

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A R T I C L E   I N F O

Article history:
Received 23 May 2017
Accepted 11 June 2017
Available online 23 June 2017

MSC 2010:
60F10
33E12
60G22

Keywords:
Large deviations
Moderate deviations
Weighted Poisson distribution
First kind error probability

A B S T R A C T

A multivariate fractional Poisson process was recently defined in Beghin and Macci (2016) by considering a common independent random time change for a finite dimensional vector of independent (non-fractional) Poisson processes; moreover it was proved that, for each fixed $t \geq 0$, it has a suitable multinomial conditional distribution of the components given their sum. In this paper we consider another multivariate process $\{\underline{M}^{\nu}(t) = (M_1^{\nu}(t), \ldots, M_m^{\nu}(t)) : t \geq 0\}$ with the same conditional distributions of the components given their sums, and different marginal distributions of the sums; more precisely we assume that the one-dimensional marginal distributions of the process $\left\{\sum_{i=1}^{m} M_i^{\nu}(t) : t \geq 0\right\}$ coincide with the ones of the alternative fractional (univariate) Poisson process in Beghin and Macci (2013). We present large deviation results for $\{\underline{M}^{\nu}(t) = (M_1^{\nu}(t), \ldots, M_m^{\nu}(t)) : t \geq 0\}$, and this generalizes the result in Beghin and Macci (2013) concerning the univariate case. We also study moderate deviations and we present some statistical applications concerning the estimation of the fractional parameter ν.

1. Introduction

Fractional Poisson processes are widely studied in the literature by considering a version of some known equations for the probability mass functions with fractional derivatives and/or fractional difference operators (see Laskin, 2003; Mainardi et al., 2004; Beghin and Orsingher, 2009; Beghin and Orsingher, 2010; Orsingher and Polito, 2012; Politi et al., 2011 and Repin and Saichev, 2000). Typically these processes are often represented in terms of randomly time-changed and subordinated processes (see e.g. Kumar et al., 2011 and Meerschaert et al., 2011) and appear in several applications (see e.g. Biard and Saussereau, 2014, where the surplus process of an insurance company is modeled by a compound fractional Poisson process).

A multivariate (space and/or time) fractional Poisson process was recently defined in Beghin and Macci (2016) by considering a common independent random time change in terms of the stable subordinator and/or its inverse for a finite dimensional vector of independent (non-fractional) Poisson processes. In the proof of Proposition 4 in Beghin and Macci (2016) it was proved that, for each fixed $t \geq 0$, the conditional (joint) distribution of the components of this multivariate process given their sum is multinomial; moreover this conditional multinomial distribution does not depend on t and on the fractional parameters.

In this paper we consider another multivariate process $\{\underline{M}^{\nu}(t) = (M_1^{\nu}(t), \ldots, M_m^{\nu}(t)) : t \geq 0\}$ with the same conditional distributions of the components given their sums, but we change the distribution of the sums of the components. More

☆ The authors acknowledge the support of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
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http://dx.doi.org/10.1016/j.spl.2017.06.009

precisely we assume that the one-dimensional marginal distributions of the process $\left\{\sum_{i=1}^{m} M_i^{\nu}(t) : t \geq 0\right\}$ coincide with the ones of the alternative fractional (univariate) Poisson process in Beghin and Macci (2013); in other words we mean the alternative fractional Poisson processes in Beghin and Orsingher (2009) with a deterministic time-change. Thus it is natural to define the process in this paper as the multivariate version of the alternative fractional Poisson process.

The alternative fractional Poisson process in Beghin and Orsingher (2009) appears as the process which counts the number of changes of direction of a fractional telegraph process (see e.g. (4.7) in Garra et al., 2014), and of a reflected random flight on the surface of a sphere (see e.g. (4.24) in De Gregorio and Orsingher, 2015). Some generalizations of the alternative fractional Poisson process in Beghin and Orsingher (2009) can be found in Garra et al. (2015) (see (3.5)) and in Pogány and Tomovski (2016) (see Proposition 2.1). In all these cases we have a weighted Poisson process as in Balakrishnan and Kozubowski (2008); the concept of weighted Poisson process for $\left\{\sum_{i=1}^{m} M_i^{\nu}(t) : t \geq 0\right\}$ is illustrated in Remark 1.

The aim of this paper is to present large deviation results for the multivariate version of the alternative fractional Poisson process. The theory of large deviations gives an asymptotic computation of small probabilities on exponential scale (see e.g. Dembo and Zeitouni, 1998 as references on this topic). The main results in this paper are Propositions 1 and 2, which concern large and moderate deviations. The main tool used in the proofs of Propositions 1 and 2 is the Gärtner Ellis Theorem (see e.g. Section 2.3 in Dembo and Zeitouni, 1998). We point out that in Beghin and Macci (2013) we study large deviations only; in particular Proposition 1 in this paper reduces to Proposition 4.1 in Beghin and Macci (2013) if we consider the univariate case $m = 1$ (see Remark 2).

The term moderate deviations is used for a class of large deviation principles governed by the same quadratic rate function which uniquely vanishes at the origin. Typically moderate deviations fill the gap between a convergence to zero and an asymptotic Normality result. We also recall that, as pointed out in some references (see e.g. Bryc, 1993 and the references cited therein), under certain conditions one can obtain the weak convergence to a centered Normal distribution whose variance is determined by a large deviation principle obtained by the Gärtner Ellis Theorem.

We conclude with the outline of the paper. We start with some preliminaries in Section 2. The multivariate process studied in this paper is defined in Section 3. Large and moderate deviation results are presented in Section 4. We conclude with some statistical applications in Section 5.

2. Preliminaries

We always set $0 \log 0 = 0$. In general we deal with vectors in $\mathbb{R}^m$ and we use the following notation: $\underline{x} = (x_1, \ldots, x_m)$, and $\underline{0} = (0, \ldots, 0)$ is the null vector; $\underline{x} \geq \underline{0}$ means that $x_1, \ldots, x_m \geq 0$; we set $s(\underline{x}) = \sum_{i=1}^{m} x_i$ and $\langle \underline{x}, \underline{y} \rangle = \sum_{i=1}^{m} x_i y_i$.

2.1. Preliminaries on large (and moderate) deviations

We recall the basic definitions (see e.g. Dembo and Zeitouni, 1998, pages 4–5). Let $\mathcal{Z}$ be a Hausdorff topological space with Borel σ-algebra $\mathcal{B}_{\mathcal{Z}}$. A speed function is a family of numbers $\{v_t : t > 0\}$ such that $\lim_{t\to\infty} v_t = \infty$. A lower semi-continuous function $I : \mathcal{Z} \to [0, \infty]$ is called rate function. A family of $\mathcal{Z}$-valued random variables $\{Z_t : t > 0\}$ satisfies the *large deviation principle* (LDP for short), as $t \to \infty$, with speed function v_t and rate function I if

$$\limsup_{t\to\infty} \frac{1}{v_t} \log P(Z_t \in F) \leq -\inf_{z\in F} I(z) \quad \text{(for all closed sets } F)$$

and

$$\liminf_{t\to\infty} \frac{1}{v_t} \log P(Z_t \in G) \geq -\inf_{z\in G} I(z) \quad \text{(for all open sets } G).$$

A rate function I is said to be good if all the level sets $\{\{z \in \mathcal{Z} : I(z) \leq \gamma\} : \gamma \geq 0\}$ are compact.

The term *moderate deviations* is used when, for all positive numbers $\{a_t : t > 0\}$ such that

$$a_t \to 0 \quad \text{and} \quad ta_t \to \infty \quad (\text{as } t \to \infty), \tag{1}$$

we have a LDP for suitable centered random variables on $\mathcal{Z} = \mathbb{R}^m$ (for some $m \geq 1$) with speed $1/a_t$ and the same quadratic rate function which uniquely vanishes at the origin of $\mathbb{R}^m$ (we mean that the rate function does not depend on the choice of $\{a_t : t > 0\}$). Typically moderate deviations fill the gap between two regimes (for the second one see Remark 7):

- a convergence (at least in probability) to zero of centered random variables (case $a_t = \frac{1}{t}$);
- a weak convergence to a centered Normal distribution (case $a_t = 1$).

Note that in both cases one condition in (1) fails.