



A note on the asymptotics of the maxima for the St. Petersburg game

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ABSTRACT

In this note, we consider the maxima of payoffs for the generalized St. Petersburg game. The maxima for the original St. Petersburg game cannot be normalized to converge to a nondegenerate limit distribution. However, tuning the parameters appearing in the generalization, we show the normalized maxima converge to the Fréchet distribution.

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1. Introduction and results

1.1. Notation

For real sequences $\{a_n\}$ and $\{b_n\} \subset \mathbb{R}$ symbols $a_n \sim b_n$, $a_n \asymp b_n$, $a_n = o(b_n)$ and $a_n = O(b_n)$ stand for $\lim a_n/b_n = 1$, $0 < \liminf a_n/b_n \leq \limsup a_n/b_n < \infty$, $\lim a_n/b_n = 0$ and $\limsup a_n/b_n < \infty$, respectively. Moreover, symbols $\lfloor x \rfloor$, $\lceil x \rceil$ and $\langle x \rangle$ for $x \in \mathbb{R}$ denote the integer part $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$, the upper integer part $\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$ and the fractional part $\langle x \rangle = x - \lfloor x \rfloor$, respectively.

1.2. Background

Let us consider that a fair coin is tossed repeatedly until it falls heads. If this happens at the k th trial then a player receives 2^k yen for $k = 1, 2, \dots$. Since the distribution of the payoff X is

$$P(X = 2^k) = 2^{-k} \quad \text{for } k = 1, 2, \dots, \quad (1)$$

it turns out that $E(X) = \infty$, which is called the *St. Petersburg game* (see Chapter X.4 in [Feller, 1968](#)). There exist several generalizations of this game. For example, [Gut and Martin-Löf \(2015\)](#) studied the following distribution

$$P(X = sr^{k-1}) = pq^{k-1} \quad \text{for } k = 1, 2, \dots, \quad (2)$$

where

$$s, r > 0, \quad 0 < p = 1 - q < 1 \quad \text{and} \quad rq \geq 1. \quad (3)$$

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Indeed, when $s = r = 2$ and $p = q = 1/2$ hold, Eq. (2) is equivalent to Eq. (1). Note that the last condition of Eq. (3) guarantees $E(X) = \infty$. Let $X_1, X_2, \dots, X_n$ be the payoffs for n times repetitions of the original St. Petersburg game. Namely, they are independent and identically distributed (i.i.d.) random variables with the common distribution determined by Eq. (1). For the total payoffs $S_n := \sum_{i=1}^n X_i$, it is written that $\lim_{n \rightarrow \infty} S_n / (n \log_2 n) = 1$ in probability in Chapter X of Feller (1968). However, the distribution for Eq. (1) does not belong to the domain of attraction of any stable distributions. Martin-Löf (1985) showed that $S_n/n - \log_2 n$ converges to a semi-stable distribution along the subsequence $n = 2^k$. Based on this result, Csörgő (2002) studied a *merge convergence* for S_n . The asymptotics of S_n is not the main topic of this note, so we do not touch any more on this here.

Instead of S_n , the maximum for n payoffs

$$M_n := \max\{X_1, X_2, \dots, X_n\} \quad (4)$$

is also studied. The distribution for Eq. (1) is not in the maximum domain of attraction of any max-stable law. In fact, it follows by the reason that the distribution for Eq. (1) does not satisfy Equation (1.7.4) of Leadbetter et al. (1983). Therefore, some ideas are necessary for convergence of M_n in some sense. For example, Equation (4) of Berkes et al. (1999) tells

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{M_n}{n} \leq x\right) - H_{\gamma_n}(x) \right| = O(1/n), \quad (5)$$

where

$$\gamma_n := n/2^{\lceil \log_2 n \rceil} \quad \text{and} \quad H_{\gamma_n}(x) := \begin{cases} 0 & \text{for } x \leq 0 \\ e^{-\gamma_n 2^{-\lfloor \gamma_n x \rfloor}} & \text{for } x > 0. \end{cases}$$

It seems to be the maxima version of the merge convergence. On the other hand, Equation (2.6) of Gut and Martin-Löf (2016) indicates

$$P(M(t) \leq 2^m) = e^{-t2^{-m}} \quad \text{for } m = 1, 2, \dots, \quad (6)$$

where $M(t) := \lim_{n \rightarrow \infty} M_{t2^n}/2^n$ for $t > 0$. As continuation of Eq. (6), Gut and Martin-Löf (2016) studied the *maxtrimmed* sum, namely S_n minus all maximal payoffs, by computing characteristic functions. Berkes et al. (in press) also investigated a trimmed sum paying attention to the subexponential property.

Moreover, let us remark that Stoica (2008) gave logarithmic tail asymptotics for not only S_n but also M_n while avoiding this difficulty.

1.3. Our contribution

In this note, we try another approach for investigating the asymptotics of the maxima for the generalized St. Petersburg game. While Examples 1.7.14 and 1.7.15 in Leadbetter et al. (1983) illustrate that no limit distributions exist for the normalized maxima of the i.i.d. random variables of the Poisson distribution and the geometric distribution, Proposition 1 of Anderson et al. (1997) and Corollary 1 of Nadarajah and Mitov (2002) provide that the normalized maxima with tuning parameters converge to the Gumbel distributions respectively.

Here, when investigating M_n for the generalized St. Petersburg game, parameters p, q and r appearing in Eq. (2) will be tuned in the following theorem.

Theorem 1.1. Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables with the common distribution determined by Eqs. (2) and (3). We assume that $p = p(n)$, $q = q(n)$ and $r = r(n)$ depend on n and

$$p(n) = 1 - q(n) = o(1). \quad (7)$$

Moreover, there exists $\alpha > 0$ satisfying

$$\alpha = - \lim_{n \rightarrow \infty} \frac{\log q(n)}{\log r(n)}. \quad (8)$$

Then we have

$$\lim_{n \rightarrow \infty} P(M_n \leq a(n)x) = \Phi_\alpha(x), \quad (9)$$

where $\Phi_\alpha(x)$ is the Fréchet distribution

$$\Phi_\alpha(x) := \begin{cases} e^{-x^{-\alpha}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0, \end{cases} \quad (10)$$

and

$$a(n) = sn^{1/\alpha}. \quad (11)$$

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