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The left tail of renewal measure

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ABSTRACT

In the paper, we find exact asymptotics of the left tail of renewal measure for a broad class of two-sided random walks. We only require that an exponential moment of the left tail is finite. Through a simple change of measure approach, our result turns out to be almost equivalent to Blackwell's Theorem.

1. Introduction

Let $(X_k)_{k\geq 1}$ be a sequence of independent copies of a random variable X with $\mathbb{E}X > 0$ (we allow $\mathbb{E}X = \infty$). Further, define $S_n = X_1 + \cdots + X_n$, $n \geq 1$ and $S_0 = 0$. The measure defined by

$$H(B) := \sum_{n=0}^{\infty} \mathbb{P}(S_n \in B), \quad B \in \mathcal{B}(\mathbb{R})$$

is called the *renewal measure of* $(S_n)_{n\geq 1}$.

We say that the distribution of a random variable X is *d-arithmetic* ($d > 0$) if it is concentrated on $d\mathbb{Z}$ and not concentrated on $d'\mathbb{Z}$ for any $d' > d$. A distribution is said to be *non-arithmetic* if it is not d-arithmetic for any $d > 0$.

A fundamental result of renewal theory is the Blackwell Theorem (Blackwell, 1953): if the distribution of X is non-arithmetic, then for any $h > 0$,

$$H\left((x, x+h]\right) \longrightarrow \frac{h}{\mathbb{E}X} \quad \text{as } x \to \infty. \tag{1}$$

If the distribution of X is d-arithmetic, then for any $h > 0$,

$$H\left((dn, dn+h]\right) \longrightarrow \frac{d\lfloor h/d \rfloor}{\mathbb{E}X} \quad \text{as } n \to \infty. \tag{2}$$

The above results remain true if $\mathbb{E}X = \infty$ with the usual convention that $c/\infty = 0$ for any finite c. In the infinite-mean case the exact asymptotics of $H((x, x+h])$ are also known. Assume that X is a non-negative random variable with a non-arithmetic law such that $\mathbb{P}(X > x) = L(x)x^{-\alpha}$ with $\alpha \in (0, 1)$, where L is a slowly varying function. Then $\mathbb{E}X = \infty$. If $\alpha \in (1/2, 1)$, then without additional assumptions the so called Strong Renewal Theorem holds, for $h > 0$,

$$m(x)H\left((x, x+h]\right) \longrightarrow \frac{h}{\Gamma(\alpha)\Gamma(2-\alpha)} \quad \text{as } x \to \infty, \tag{3}$$

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where $m(x) = \int_0^x \mathbb{P}(X > t)\mathrm{d}t \sim L(x)x^{1-\alpha}/(1-\alpha) \to \infty$. Here and later on $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$.

The case of $\alpha \in (0, 1/2]$ is much harder and was completely solved just recently by Caravenna and Doney (2016). It was shown that if $\alpha \in (0, 1/2]$ and X is a non-negative random variable with regularly varying tail, then (3) holds if and only if (Caravenna and Doney, 2016 Proposition 1.11)

$$\lim_{\delta\to 0}\limsup_{x\to\infty}\int_1^{\delta x}\frac{F(x)-F(x-z)}{\overline{F}(z)z^2}\mathrm{d}z = 0, \tag{4}$$

where F is the cumulative distribution function of X and $\overline{F} = 1 - F$. It was already observed by Kevei (2016, Theorem 3.1) that this result generalizes to X attaining negative values as well if additionally

$$\mathbb{P}(X \le -x) = o(e^{-rx}) \quad \text{as } x \to \infty \tag{5}$$

for some $r > 0$. This will be our setup. Full picture of SRT for random walks is also known (Caravenna and Doney, 2016 Theorem 1.12).

It is clear that $\lim_{x\to\infty} H((-\infty, -x)) = 0$. There are considerably fewer papers dedicated to analysis of exact asymptotics of such object than of $H((x, x+h])$ as in Blackwell's Theorem. Under some additional assumptions we know more about the asymptotic behaviour of the left tail. Stone (1965) proved that if for some $r > 0$ (5) holds, then for some $r_1 > 0$,

$$H((-\infty, -x)) = o(e^{-r_1 x}) \quad \text{as } x \to \infty. \tag{6}$$

Stone's result was strengthened by van der Genugten (1969), where exact asymptotics as well the speed of convergence of the remainder term are given for d-arithmetic and spread-out laws (i.e. laws, whose nth convolution has a nontrivial absolutely continuous part for some $n \in \mathbb{N}$). An important contribution regarding the asymptotics of the left tail of renewal measure was made by Carlsson (1983), who concerned with the case when $\mathbb{E}|X|^m < \infty$ for some $m \ge 2$, but this does not fit well into our setup. We allow $\mathbb{E}X_+ = \infty$, but on the other hand we require that some exponential moments of X_- exist. The results mentioned above were obtained using some analytical methods, whereas we will use a simple probabilistic argument, which boils down the asymptotics of $H((-\infty, -x))$ to the asymptotics of $\widetilde{H}((x, x+h])$, where $\widetilde{H}$ is some new (possibly defective, see below) renewal measure.

1.1. Defective renewal measure

For $\rho \in (0, 1)$ consider

$$H_\rho(B) := \sum_{n=0}^{\infty}\rho^n \mathbb{P}(S_n \in B), \quad B \in \mathcal{B}(\mathbb{R}),$$

where $(S_n)_{n\ge 1}$ is, as in the previous section, a random walk starting from 0. H_ρ is called a *defective renewal measure of* $(S_n)_{n\ge1}$. In contrast to the renewal measure, H_ρ is a finite measure. Let τ be independent of $(S_n)_{n\ge1}$ and $\mathbb{P}(\tau = n) = (1-\rho)\rho^n$, $n = 0, 1, \ldots$. Then $H_\rho(B) = \mathbb{P}(S_\tau \in B)/(1-\rho)$. It is well known that if the distribution of S_1 is subexponential, then $\mathbb{P}(S_\tau > x) \sim \mathbb{E}\tau\,\mathbb{P}(S_1 > x)$. Here, we are interested in exact asymptotics of $H_\rho(B)$ when $B = (x, x+T]$ for any $T > 0$. In this context, local subexponentiality is the key concept (Asmussen et al., 2003).

Let μ be a probability measure on $\mathbb{R}$. For $T > 0$ we write $\Delta = (0, T]$ and $x + \Delta = (x, x+T]$. We say that μ belongs to the class $\mathcal{L}_\Delta$ if $\mu(x+\Delta) > 0$ for sufficiently large x and

$$\frac{\mu(x+s+\Delta)}{\mu(x+\Delta)} \to 1 \quad \text{as } x \to \infty, \tag{7}$$

uniformly in $s \in [0, 1]$.

We say that μ is Δ-subexponential if $F \in \mathcal{L}_\Delta$ and

$$\mu^{*2}(x+\Delta) \sim 2\mu(x+\Delta).$$

Then we write $\mu \in \mathcal{S}_\Delta$. Finally, μ is called *locally subexponential* if $\mu \in \mathcal{S}_\Delta$ for any $T > 0$. We denote this class by $\mathcal{S}_{loc}$.

The following theorem is an obvious conclusion from Watanabe and Yamamuro (2009, Theorem 1.1).

Theorem 1.1. *Assume that* μ *is a probability measure on* $\mathbb{R}$ *such that*

$$\int_{\mathbb{R}} e^{-\varepsilon x}\mu(\mathrm{d}x) < \infty \quad \textit{for some } \varepsilon > 0.$$

For $0 < \rho < 1$ *define*

$$\eta = \sum_{n=0}^{\infty}\rho^n\mu^{*n}.$$