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Factorial designs robust against the presence of an aberration

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ABSTRACT

In this paper, designs, insensitive to the presence of an outlier for estimating a complete set of orthonormal treatment contrasts representing the factorial effects, are obtained for both complete and fractional factorial designs. Fractional factorials incorporated in blocks are also considered in this study. In case of blocked complete factorials, semi-regular group divisible designs and for fractional factorials, universally optimal plans are found to be robust.

1. Introduction

Factorial experiments have wide applications in many areas of research. Factorial designs were introduced and popularized by Fisher (1935). The early authors were Bose and Kishen (1940), Bose (1947), Nair and Rao (1948), Shah (1958) and some more. A comprehensive discussion on complete factorials and that on fractional designs can be obtained in Gupta and Mukerjee (1989) and Dey and Mukerjee (1999) respectively.

An outlying observation is one which does not fit in with the pattern of the remaining observations in a data set. The problem here is to find factorial designs insensitive to the presence of such an outlier.

John (1978) discussed methods of detecting outliers in the factorial setup. Ghosh and Kipngeno (1985) studied the sensitivity of the optimum balanced resolution V plans for 2^{m} factorials to outliers, using the measure suggested by Box and Draper (1975), and observed that such designs are robust.

In the present article an attempt has been made to study the robustness of factorial designs for the estimation of a complete set of orthonormal treatment contrasts, $\boldsymbol{\psi}$, representing factorial effects, against the presence of an outlier. Complete factorials in blocks and fractional factorials without and with blocks are studied in this respect.

The article is organized as follows:

In Section 2, the parameters of interest in a factorial design, viz. the main effects and interactions, are introduced. In Sections 3 and 4, robustness of complete factorials and fractional factorials are respectively studied. Section 5 contains robustness of fractional factorials in blocks.

2. The factorial setup

Consider a factorial experiment involving n factors, $F_1, F_2, \ldots, F_n$, at $m_1, m_2, \ldots, m_n$ (≥ 2) levels respectively. Let the levels of F_i be coded as $0, 1, \ldots, m_i - 1\,(1 \leq i \leq n)$. Let $\Omega = \{\mathbf{x} = (x_1, x_2, .., x_n) : x_i = 0, 1 \forall i = 1, 2, \ldots, n, (x_1, x_2, \ldots, x_n) \neq$

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$(0, 0, \ldots, 0)\}$ be the set of all n-component non-null binary vectors. It is easy to see that there is a one-to-one correspondence between Ω and the set of all interactions (cf. Gupta and Mukerjee, 1989), in the sense that a typical interaction $F_{i_1}F_{i_2}\ldots F_{i_g}$ $(1 \le i_1 < i_2 < \cdots < i_g \le n, 1 \le g \le n)$ corresponds to the element $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$ of Ω such that $x_{i_1} = x_{i_2} = \cdots = x_{i_g} = 1$ and $x_u = 0$ for $u \neq i_1, i_2, \ldots, i_g$. Thus, the $2^n - 1$ interactions may be denoted by $F^{\mathbf{x}}$, $\mathbf{x} \in \Omega$. A typical selection of levels $j = (j_1, j_2, \ldots, j_n)$, $0 \le j_i \le m_i - 1$, $1 \le i \le n$, will be termed the jth treatment combination and the effect due to this treatment combination will be denoted by $\tau(j_1, j_2, \ldots, j_n)$. τ is the vector of all possible level combinations. Now, the treatment contrasts belonging to the interactions are conveniently represented making use of Kronecker products, as indicated below.

For each $\mathbf{x} \in \Omega$, let

$$\mathbf{M}^{\mathbf{x}} = \mathbf{M}_1^{x_1} \otimes \mathbf{M}_2^{x_2} \otimes \cdots \otimes \mathbf{M}_n^{x_n} = \bigotimes_{i=1}^{n} \mathbf{M}_i^{x_i}, \tag{2.1}$$

where $\otimes$ denotes Kronecker product and for $1 \le i \le n$,

$$\begin{aligned} \mathbf{M}_i^{x_i} &= \mathbf{I}_i - m_i^{-1}\mathbf{J}_i \quad \text{if } x_i = 1 \\ &= m_i^{-1}\mathbf{J}_i \qquad\quad \text{if } x_i = 0. \end{aligned} \tag{2.2}$$

Here $\mathbf{I}_i$ is the identity matrix and $\mathbf{J}_i$ is the matrix of 1s, both of order $m_i \times m_i$. For each $\mathbf{x} \in \Omega$, the elements of $\mathbf{M}^{\mathbf{x}}\tau$ represent a complete set of treatment contrasts belonging to the interaction $F^{\mathbf{x}}$ (cf. Gupta and Mukerjee, 1989).

Another equivalent representation may be given in terms of orthonormal contrasts. For $1 \le i \le n$, let $\mathbf{1}_i$ be the $m_i \times 1$ vector with all elements unity and $\mathbf{P}_i$ be an $(m_i - 1) \times m_i$ matrix such that the $m_i \times m_i$ matrix $(m_i^{-1/2}\mathbf{1}_i, \mathbf{P}_i')'$ is orthogonal. For each $\mathbf{x} \in \Omega$, let

$$\mathbf{P}^{\mathbf{x}} = \bigotimes_{i=1}^{n} \mathbf{P}_i^{x_i}, \tag{2.3}$$

where for $1 \le i \le n$,

$$\begin{aligned} \mathbf{P}_i^{x_i} &= \mathbf{P}_i \qquad\qquad \text{if } x_i = 1 \\ &= m_i^{-1/2}\mathbf{1}_i' \quad \text{if } x_i = 0. \end{aligned} \tag{2.4}$$

For every i, it follows that

$$\mathbf{P}_i^{x_i\prime}\mathbf{P}_i^{x_i} = \mathbf{M}_i^{x_i} \tag{2.5}$$

and that for every $\mathbf{x} \in \Omega$,

$$\mathbf{P}^{\mathbf{x}\prime}\mathbf{P}^{\mathbf{x}} = \mathbf{M}^{\mathbf{x}}. \tag{2.6}$$

Note that for every $\mathbf{x}, \mathbf{y} \in \Omega$ and $\mathbf{x} \neq \mathbf{y}$,

$$\mathbf{P}^{\mathbf{x}}\mathbf{P}^{\mathbf{x}\prime} = \mathbf{I}, \qquad \mathbf{P}^{\mathbf{x}}\mathbf{P}^{\mathbf{y}\prime} = \mathbf{0}, \tag{2.7}$$

where $\mathbf{0}$ is the null matrix of appropriate order. Using the above facts, the factorial effects can be represented as follows (cf. Gupta and Mukerjee, 1989, Page 7).

Result 2.1. *For each $\mathbf{x} \in \Omega$, $\mathbf{P}^{\mathbf{x}}\tau$ represents a complete set of orthonormal treatment contrasts belonging to the interaction $F^{\mathbf{x}}$.*

3. Robustness of complete factorial designs in blocks

A factorial experiment is called a complete factorial when each treatment combination is applied to at least one experimental unit. In this section, we try to find robust designs in the class of complete factorial experiments.

In the block design setup, let there be b blocks of sizes k each and v treatments, $1, 2, \ldots, v$. Let $\mathbf{D}_1^{v\times n}$, $\mathbf{D}_2^{b\times n}$ be the incidence matrices of treatments vs. observations and blocks vs. observations respectively and $\mathbf{N}^{v\times b} = \mathbf{D}_1\mathbf{D}_2' = ((n_{ij}))$ be the corresponding treatments vs. blocks incidence matrix with elements n_{ij}, denoting the number of times the ith treatment occurs in the jth block, $i = 1, \ldots, v$ and $j = 1, 2, \ldots, b$. $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{1}$ are respectively the identity matrix, the matrix with all elements 1 and the vector of ones, of suitable orders.

Consider the fixed effects additive model

$$\mathbf{Y} = \mu\mathbf{1} + \mathbf{D}_1'\boldsymbol{\tau} + \mathbf{D}_2'\boldsymbol{\beta} + \mathbf{e}, \tag{3.1}$$

where $\mathbf{e}$ is the random error vector with $E(\mathbf{e}) = \mathbf{0}$ and $D(\mathbf{e}) = \sigma^2\mathbf{I}$, μ is an additive constant, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_v)'$ is the vector of unknown treatment effects and $\boldsymbol{\beta}$ is the vector of unknown block effects. If $\mathbf{C}$ denotes the characteristic matrix of the