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Asymptotic distribution of rewards accumulated by alternating renewal processes

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ABSTRACT

Consider an alternating renewal process on the states 'broken' vs 'working'. Suppose that during any interval $[0, \tau]$, the process is rewarded at rate $g(t/\tau)$ if it is working at time t. Let Q_τ be the reward that is accumulated during $[0, \tau]$. We calculate μ_{Q_τ} and $\sigma^2_{Q_\tau}$ such that $(Q_\tau - \mu_{Q_\tau})/\sigma_{Q_\tau}$ converges in distribution to a standard normal distribution as $\tau \to \infty$.

1. Introduction

Let $\mathcal{N}(\mu, \sigma^2)$ be the normal distribution with mean μ and variance σ^2, and $\Rightarrow$ denote convergence in distribution. We prove the following:

Theorem. *Let X_t be an alternating renewal process on $\{0, 1\}$ with 0 = 'broken', 1= 'working', formed from durations working $\{W_k\}$ alternated with durations broken $\{B_k\}$. Recall that there exist $z_1(t)$ and $z_0(t)$ such that*

$$\mathcal{P}\{X_t = 1|X_s = 1\} = p + (1-p)\cdot z_1(t-s)$$
$$\mathcal{P}\{X_t = 0|X_s = 0\} = 1 - p + p\cdot z_0(t-s)$$

where $p = \frac{\beta}{\alpha+\beta}$ given $\beta = \mathbb{E}(W_k)$, $\alpha = \mathbb{E}(B_k)$. Given $g : [0, 1] \to \mathbb{R}$, put $Q_\tau = \int_0^\tau g(t/\tau)X_t\,dt$ (reward the process at rate $g(t/\tau)$ if it is working at time t), and set

$$\mu_{Q_\tau} = \bar{g}\mu_{U_\tau} \qquad \mu_{U_\tau} = p\tau$$
$$\sigma^2_{Q_\tau} = \gamma\sigma^2_{U_\tau} \qquad \sigma^2_{U_\tau} = 2p(1-p)\tau\zeta$$

where $\bar{g} = \int_0^1 g(x)\,dx$, $\gamma = \int_0^1 (g(x))^2 dx$, $\zeta = \int_0^\infty z(t)\,dt$, and $z(t) = (1-p)\cdot z_1(t) + p\cdot z_0(t)$. Suppose that all of the following conditions are satisfied:

- $\mathbb{E}(W_k^2) + \mathbb{E}(B_k^2) > 0$, $\mathbb{E}(W_k^3) < \infty$, $\mathbb{E}(B_k^3) < \infty$, *for all k.*

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- $0 < \zeta < \infty$, *and there exists* $\hat{z}(t)$ *continuous and nonincreasing such that* $|z(t)| \le \hat{z}(t)$ *for all t sufficiently large and* $\int_0^\infty \hat{z}(t)\,dt < \infty$.
- $-\infty < \bar{g} < \infty$, $0 < \gamma < \infty$, *and* $|\int_0^1 g(x)g'(x)\,dx| < \infty$.

Then $(Q_\tau - \mu_{Q_\tau})/\sigma_{Q_\tau} \Rightarrow \mathcal{N}(0, 1)$ *as* $\tau \to \infty$.

Remark. If $F(x) = g^{-1}(x)$ is a well-defined cumulative distribution function, and μ_R and σ_R^2 are the mean and variance of the distribution defined by F, then $\bar{g} = \mu_R$ and $\gamma = \sigma_R^2 + \mu_R^2$.

The finding appears to be novel in studies of alternating renewal processes, in two respects: First, the process accumulates a reward at rate g. Second, the value obtained for $\sigma_{U_\tau}^2$ is new. Indeed, we see that $\sigma_{U_\tau}^2$ is fully determined by p and ζ, where ζ comes from the process forgetting its initial conditions.

Note that $W_k, B_k > 0$ for all k by definition of alternating renewal processes. The existence of z_1 and z_0 is also assured, as it is well-known (Trivedi, 2002) that X_t becomes stationary from any starting condition. While it may be difficult to explicitly obtain z_1 and z_0, we can harness a classic result by Takács (1959, Example 1): If $\sigma_\alpha^2 = \mathbb{E}(B_k^2)$, $\sigma_\beta^2 = \mathbb{E}(W_k^2)$ then $U_\tau \Rightarrow \mathcal{N}(\mu_{U_\tau}, \sigma_{U_\tau}^2)$ as $\tau \to \infty$, where

$$\mu_{U_\tau} = \frac{\beta}{\alpha + \beta}\tau$$

$$\sigma_{U_\tau}^2 = \frac{\alpha^2\sigma_\alpha^2 + \beta^2\sigma_\beta^2}{(\alpha+\beta)^3}.\tau$$

(While Takács took $\mathbb{E}(B_k^2), \mathbb{E}(W_k^2) < \infty$, this article needs $\mathbb{E}(B_k^3), \mathbb{E}(W_k^3) < \infty$.)

2. Proof

On any interval $[0, \tau]$ declare $V_\tau = Q_\tau - \bar{g}p\tau$. For any $\delta t > 0$ define $t_k = (k-1)\cdot\delta t$ and $Y_k = X_{t_k} - p$ where $k = 1, 2, \ldots$. For any positive integer n put

$$Q_{n,\delta t} = \left(g(\tfrac{1}{n})X_{t_1} + g(\tfrac{2}{n})X_{t_2} + \cdots + g(1)X_{t_n}\right)\cdot\delta t$$

$$V_{n,\delta t} = \left(g(\tfrac{1}{n})Y_1 + g(\tfrac{2}{n})Y_2 + \cdots + g(1)Y_n\right)\cdot\delta t$$

$$\sigma_{n,\delta t}^2 = (n\cdot\delta t)\cdot p\,(1-p)\cdot\left(\delta t + 2\sum_{k=1}^{\infty} z(t_k)\,\delta t\right)$$

$$\gamma_n = \frac{1}{n}\sum_{k=1}^{n}\left(g(\tfrac{k}{n})\right)^2.$$

Without loss of generality, we assume that the process is strictly stationary at time zero. For there exists s such that $z_1(s)$ and $z_0(s)$ are arbitrarily close to zero, so we may shift our analysis from $[0, \tau]$ to $[s, s+\tau]$. Shifting τ to $s+\tau$ will not matter, as we will be taking $\tau \to \infty$. Consequently Y_k is strictly stationary for all k. Moreover for all t we have $\mathcal{P}\{X_t = 1\} = p$ and $\mathcal{P}\{X_t = 0\} = 1 - p$, so $\mathbb{E}(Y_k) = 0$ for all k. Declare the following cumulative distribution functions

$$G_\tau(v) = \mathcal{P}\{V_\tau \le v\}$$

$$G_{n,\delta t}(v) = \mathcal{P}\{V_{n,\delta t} \le v\}$$

$$H\left(\cdot;\mu,\sigma^2\right) \text{ for } \mathcal{N}(\mu,\sigma^2).$$

Let $\mathbb{R}$ denote the real numbers and $\mathbb{Z}_{\ge 0}$ denote the non-negative integers. We will prove the following propositions.

Proposition 1. *If* $-\infty < \bar{g} < \infty$, *then for any* $v \in \mathbb{R}$, $\psi > 0$, *and* $\epsilon_1 > 0$ *there exists* $\delta t_1 > 0$ *such that if* $\delta t < \delta t_1$, $m = \left\lfloor \frac{\psi}{\delta t} \right\rfloor$, $n = m + m'$ *for any* $m' \in \mathbb{Z}_{\ge 0}$, *and* $\tau = n\cdot\delta t$ *then* $\left|G_{n,\delta t}(v) - G_\tau(v)\right| < \epsilon_1$.

Proposition 2. *If* $0 < p < 1$, $0 < \zeta < \infty$, *and* $0 < \gamma < \infty$, *then for any* $v \in \mathbb{R}$, $\psi > 0$, *and* $\epsilon_2 > 0$ *there exists* $\delta t_2 > 0$ *such that if* $\delta t < \delta t_2$, $m = \left\lfloor \frac{\psi}{\delta t} \right\rfloor$, $n = m + m'$ *for any* $m' \in \mathbb{Z}_{\ge 0}$ *and* $\tau = n\cdot\delta t$ *then* $0 < \sigma_{n,\delta t}^2 < \infty$ *and* $\left|H\left(v; 0, \gamma_n\sigma_{n,\delta t}^2\right) - H\left(v; 0, \sigma_{Q_\tau}^2\right)\right| < \epsilon_2$.

Proposition 3. *If* $\mathbb{E}(W_k^2) + \mathbb{E}(B_k^2) > 0$, $\mathbb{E}(W_k^3) < \infty$, $\mathbb{E}(B_k^3) < \infty$ *for all k*, $0 < \zeta < \infty$, *there exists* $\hat{z}(t)$ *continuous and nonincreasing such that* $|z(t)| \le \hat{z}(t)$ *for all t sufficiently large and* $\int_0^\infty \hat{z}(t)\,dt < \infty$, *and* $0 < \gamma < \infty$ *and* $|\int_0^1 g(x)g'(x)\,dx| < \infty$, *then for any* $v \in \mathbb{R}$, $\delta t > 0$, *and* $\epsilon_3 > 0$ *there exists* $N_3 > 0$ *such that if* $n > N_3$ *and* $0 < \sigma_{n,\delta t}^2 < \infty$ *then* $\left|G_{n,\delta t_3}(v) - H\left(v; 0, \gamma_n\sigma_{n,\delta t_3}^2\right)\right| < \epsilon_3$.