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Conditional Stein approximation for Itô and Skorohod integrals

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ABSTRACT

We derive conditional Edgeworth-type expansions for Skorohod and Itô integrals with respect to Brownian motion, based on cumulant operators defined by the Malliavin calculus. As a consequence we obtain conditional Stein approximation bounds for multiple stochastic integrals and quadratic Brownian functionals.

1. Introduction

Let $(B_t)_{t\in[0,T]}$ be a d-dimensional Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ on the Wiener space Ω. It has been shown in Theorem 2.1 of Driver et al. (2016), that given B_T, the stochastic integral $\int_0^T AB_s dB_s$ is Gaussian $\mathcal{N}\left(0, \int_0^T |AB_s|^2\, ds\right)$-distributed given $\int_0^T |AB_s|^2\, ds$ when the $d \times d$ matrix A is skew-symmetrix, as an extension of results of Yor (1980) in the case of Lévy's stochastic area with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. On the other hand, it has recently been shown in Privault and She (2015) that the distribution of $\int_0^T AB_s dB_s$ given $\int_0^T |AB_s|^2\, ds$ is also Gaussian $\mathcal{N}\left(0, \int_0^T |AB_s|^2\, ds\right)$ when A is a 2-nilpotent $d \times d$ matrix, in connection with results of Yor (1979) showing that the filtration $(\mathcal{F}_t^k)_{t\in[0,T]}$ of $t \mapsto \int_0^t AB_s dB_s$ is generated by k independent Brownian motions, where k is the number of distinct eigenvalues of $A^\top A$.

More generally, this type of result has been shown to hold in Privault and She (2015) for stochastic integrals of the form $\int_0^T u_t dB_t$ where $(u_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-adapted process, under conditions formulated in terms of the Malliavin calculus, based on the cumulant–moment formulas of Privault (2013, 2015a). Namely, sufficient conditions on the process $(u_t)_{t\in[0,T]}$ have been given for $\int_0^T u_t dB_t$ to be Gaussian $\mathcal{N}\left(0, \int_0^T |u_t|^2 dt\right)$-distributed given $\int_0^T |u_t|^2 dt$, cf. Theorem 2 therein.

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In this paper, using the Malliavin–Stein method on the Wiener space we derive conditional estimates on the distance between the law of $\int_0^T u_t dB_t$ and the Gaussian $\mathcal{N}\left(0, \int_0^T |u_t|^2 dt\right)$ distribution given $\int_0^T |u_t|^2 dt$. For this, we rely on conditional Edgeworth type expansions for random variables represented as the Itô stochastic integral of $(u_t)_{t\in[0,T]}$ with respect to $(B_t)_{t\in[0,T]}$, extending results of Privault (2015b) to a conditional setting. This approach relies on properties of the Skorohod integral operator δ, which coincides with the Itô stochastic integral with respect to d-dimensional Brownian motion on the square-integrable adapted processes.

Letting $H = L^2(\mathbb{R}_+; \mathbb{R}^d)$, we consider the standard Sobolev spaces of real-valued, resp. H-valued, functionals $\mathbb{D}_{p,k}$, resp. $\mathbb{D}_{p,k}(H)$, $p, k \geq 1$, for the Malliavin gradient D on the Wiener space, cf. Section 1.2 of Nualart (2006) for definitions. Recall that the Skorohod operator δ is the adjoint of the gradient D through the duality relation

$$E[F\delta(v)] = E[\langle DF, v\rangle_H], \quad F \in \mathbb{D}_{2,1},\ v \in \mathbb{D}_{2,1}(H), \tag{1.1}$$

and we have the commutation relation

$$D_t\delta(u) = u(t) + \delta(D_t u), \quad t \in \mathbb{R}_+, \tag{1.2}$$

provided that $u \in \mathbb{D}_{2,1}(H)$ and $D_t u \in \mathbb{D}_{2,1}(H)$, dt-$a.e.$, cf. Proposition 1.3.2 of Nualart (2006). In the sequel we let $\langle h, h\rangle = \langle h, h\rangle_H$ and $\|h\| = \|h\|_H$, $h \in H$.

First order conditional duality and expansion

The duality relation (1.1) shows that we have

$$E\left[F\delta(u)f(\delta(u))\right] = E\left[F\langle u, u\rangle f'(\delta(u))\right] + E\left[\langle u, DF\rangle f(\delta(u))\right] + E\left[Ff'(\delta(u))\langle u, \delta(Du)\rangle\right],$$

for $f \in \mathcal{C}_b^1(\mathbb{R})$, provided that $u \in \mathbb{D}_{2,2}(H)$. Applying this relation to $F = g(\langle u, u\rangle)$ where $g : (0, \infty) \to (0, \infty)$ is in $\mathcal{C}_b^1((0, \infty))$, under the condition $\langle u, (Du)u\rangle = 0$ we have

$$\begin{aligned}
&E\left[F\delta(u)f(\delta(u))\right] \\
&\quad = E\left[F\langle u, u\rangle f'(\delta(u))\right] + E\left[\langle u, D\langle u, u\rangle f(\delta(u))g'(\langle u, u\rangle)\right] + E\left[Ff'(\delta(u))\langle u, \delta(Du)\rangle\right] \\
&\quad = E\left[F\langle u, u\rangle f'(\delta(u))\right] + 2E\left[\langle u, (Du)u\rangle f(\delta(u))g'(\langle u, u\rangle)\right] + E\left[F\langle u, \delta(Du)\rangle f'(\delta(u))\right] \\
&\quad = E\left[F\langle u, u\rangle f'(\delta(u))\right] + E\left[F\langle u, \delta(Du)\rangle f'(\delta(u))\right],
\end{aligned}$$

which yields

$$E_{|u|}\left[\delta(u)f(\delta(u)) - \langle u, u\rangle f'(\delta(u))\right] = E_{|u|}\left[\langle u, \delta(Du)\rangle f'(\delta(u))\right], \tag{1.3}$$

for $u \in \mathbb{D}_{2,1}(H)$, $F \in \mathbb{D}_{2,1}$ and $f \in \mathcal{C}_b^1(\mathbb{R})$, where

$$E_{|u|}[F] := E[F \mid \langle u, u\rangle]$$

denotes the conditional expectation given $\langle u, u\rangle$.

Let now $\mathcal{N}(0, g(\|u\|))$ denote a centered Gaussian random variable with variance $g(\|u\|)$, where $g : (0, \infty) \to (0, \infty)$ is a measurable function. Applying the above relation (1.3) to the solution f_x of the Stein equation

$$\mathbf{1}_{(-\infty,x]}(z) - \Phi_{g(\|u\|)}(x) = g(\|u\|)f_x'(z) - zf_x(z), \quad z \in \mathbb{R}, \tag{1.4}$$

satisfying the bounds $\|f_x\|_\infty \leq \sqrt{2\pi}/4$ and $\|f_x'\|_\infty \leq 1/\sqrt{g(\|u\|)}$, cf. Lemma 2.2-(v) of Chen et al. (2011), yields the conditional expansion

$$P(\delta(u) \leq x \mid \|u\|) = \Phi_{g(\|u\|)}(x) + E_{|u|}\left[(g(\|u\|) - \langle u, u\rangle)f_x'(\delta(u))\right] - E\left[\langle u, \delta(Du)\rangle f_x'(\delta(u))\right],$$

$x \in \mathbb{R}$, around the Gaussian cumulative distribution function $\Phi_{g(\|u\|)}(x)$, with $u \in \mathbb{D}_{2,1}(H)$.

In Section 2 we will expand this approach to Edgeworth type expansion of all orders, based on a family of cumulant operators that are associated to the process $(u_t)_{t\in[0,T]}$. We refer to Barbour (1986), Nourdin and Peccati (2009a) and Biermé et al. (2012) for other approaches to Edgeworth expansions via the Stein method and the Malliavin calculus.

In Section 3, we derive conditional Stein approximation bounds for the distance between $\delta(u)$ and the Gaussian distribution with variance $g(\|u\|)$ Section 4 treats the case of double stochastic integrals, which includes the quadratic functional $\int_0^T AB_s dB_s$ as a particular case.