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Multiplying a Gaussian matrix by a Gaussian vector

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ABSTRACT

We provide a new and simple characterization of the multivariate generalized Laplace distribution. In particular, our characterization implies that the product of a Gaussian matrix with independent and identically distributed columns and an independent isotropic Gaussian vector follows a *symmetric* multivariate generalized Laplace distribution.

1. Introduction

Wishart and Bartlett (1932) proved that the inner product of two independent bidimensional standard Gaussian vectors follows a Laplace distribution. This result is deeply linked to the fact that the Laplace distribution can be represented as an infinite scale mixture of Gaussians with gamma mixing distribution. Specifically, if σ^2 follows a Gamma$(1, 1/2)$ distribution and $x|\sigma \sim \mathcal{N}(0, \sigma^2)$, then x follows a standard Laplace distribution.[1] This representation – which was recently named the *Gauss–Laplace representation* by Ding and Blitzstein (in press) following a blog post by Christian P. Robert[2] – is particularly useful if one wants to simulate a Laplace random variable: its use constitutes for example the cornerstone of the Gibbs sampling scheme for the Bayesian lasso of Park and Casella (2008).

In this short paper, we are interested in studying links between multivariate counterparts of these two characterizations. More specifically, we give a new simple characterization of the *multivariate generalized Laplace distribution* of Kotz et al. (2001). As a corollary, we show that the product of a zero-mean Gaussian matrix with independent and identically distributed (i.i.d.) columns and a zero-mean isotropic Gaussian vector follows a symmetric multivariate generalized Laplace distribution, a result that has useful applications for Bayesian principal component analysis (Bouveyron et al., 2016, 2017).

In the remainder of this paper, p and d are two positive integers and $\mathcal{S}_p^+$ denotes the cone of positive semidefinite matrices of size $p \times p$.

2. The multivariate generalized Laplace distribution

While the definition of the univariate Laplace distribution is widely undisputed, there exist several different generalizations of this distribution to higher dimensions—a comprehensive review of such generalizations can be found in the mono-

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[1] The shape-rate parametrization of the gamma distribution is used through this paper. Note also that a standard Laplace distribution is centered with variance two.

[2] https://xianblog.wordpress.com/2015/10/14/gauss-to-laplace-transmutation/.

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graph of Kotz et al. (2001). In particular, McGraw and Wagner (1968) introduced a zero-mean elliptically contoured bidimensional Laplace distribution with univariate Laplace marginals. This distribution was later generalized to the p-dimensional setting by Anderson (1992), considering characteristic functions of the form

$$\forall \mathbf{u} \in \mathbb{R}^p, \quad \phi(\mathbf{u}) = \frac{1}{1 + \frac{1}{2}\mathbf{u}^T \boldsymbol{\Sigma} \mathbf{u}},$$

where $\boldsymbol{\Sigma} \in \mathcal{S}_p^+$. This distribution was notably promoted by Eltoft et al. (2006) and is arguably the most popular multivariate generalization of the Laplace distribution (Kotz et al., 2001, p. 229). Among its advantages, this distribution can be slightly generalized to model skewness, by building on characteristic functions of the form

$$\forall \mathbf{u} \in \mathbb{R}^p, \quad \phi(\mathbf{u}) = \frac{1}{1 + \frac{1}{2}\mathbf{u}^T \boldsymbol{\Sigma} \mathbf{u} - i\boldsymbol{\mu}^T \mathbf{u}},$$

where $\boldsymbol{\mu} \in \mathbb{R}^p$ accounts for asymmetry. Similarly to the univariate Laplace distribution, this asymmetric multivariate generalization is infinitely divisible (Kotz et al., 2001, p. 256). Therefore, it can be associated with a specific Lévy process (Kyprianou, 2014, p. 5), whose increments will follow yet another generalization of the Laplace distribution, the *multivariate generalized asymmetric Laplace distribution*. This distribution, introduced by Kotz et al. (2001, p. 257) and further studied by Kozubowski et al. (2013), will be the cornerstone of our analysis of multivariate characterizations of Laplace and Gaussian distributions.

Definition 1. A random variable $\mathbf{z} \in \mathbb{R}^p$ is said to have a **multivariate generalized asymmetric Laplace distribution** with parameters $s > 0$, $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma} \in \mathcal{S}_p^+$ if its characteristic function is

$$\forall \mathbf{u} \in \mathbb{R}^p, \quad \phi_{\mathrm{GAL}_p(\boldsymbol{\Sigma},\boldsymbol{\mu},s)}(\mathbf{u}) = \left(\frac{1}{1 + \frac{1}{2}\mathbf{u}^T \boldsymbol{\Sigma} \mathbf{u} - i\boldsymbol{\mu}^T \mathbf{u}}\right)^s.$$

It is denoted by $\mathbf{z} \sim \mathrm{GAL}_p(\boldsymbol{\Sigma}, \boldsymbol{\mu}, s)$.

General properties of the generalized asymmetric Laplace distribution are discussed by Kozubowski et al. (2013). We list here a few useful ones.

Proposition 1. *Let $s > 0$, $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma} \in \mathcal{S}_p^+$. If $\mathbf{z} \sim \mathrm{GAL}_p(\boldsymbol{\Sigma}, \boldsymbol{\mu}, s)$, we have $\mathbb{E}(\mathbf{z}) = s\boldsymbol{\mu}$ and $\mathbb{C}\mathrm{ov}(\mathbf{z}) = s(\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^T)$. Moreover, if $\boldsymbol{\Sigma}$ is positive definite, the density of $\mathbf{z}$ is given by*

$$\forall \mathbf{x} \in \mathbb{R}^p, \quad f_{\mathbf{z}}(\mathbf{x}) = \frac{2e^{\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{x}}}{(2\pi)^{p/2}\Gamma(s)\sqrt{\det \boldsymbol{\Sigma}}} \left(\frac{Q_{\boldsymbol{\Sigma}}(\mathbf{x})}{C(\boldsymbol{\Sigma}, \boldsymbol{\mu})}\right)^{s-p/2} K_{s-p/2}\left(Q_{\boldsymbol{\Sigma}}(\mathbf{x})C(\boldsymbol{\Sigma}, \boldsymbol{\mu})\right),$$

where $Q_{\boldsymbol{\Sigma}}(\mathbf{x}) = \sqrt{\mathbf{x}^T \boldsymbol{\Sigma}^{-1} \mathbf{x}}$, $C(\boldsymbol{\Sigma}, \boldsymbol{\mu}) = \sqrt{2 + \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}$ and $K_{s-p/2}$ is the modified Bessel function of the second kind of order $s - p/2$.

Note that the $\mathrm{GAL}_1(2b^2, 0, 1)$ case corresponds to a centered univariate Laplace distribution with scale parameter $b > 0$. In the symmetric case ($\boldsymbol{\mu} = 0$) and when $s = 1$, we recover the multivariate generalization of the Laplace distribution of Anderson (1992).

An appealing property of the multivariate generalized Laplace distribution is that it is also endowed with a multivariate counterpart of the Gauss–Laplace representation.

Theorem 1 (*Generalized Gauss–Laplace Representation*)**.** *Let $s > 0$, $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma} \in \mathcal{S}_p^+$. If $u \sim \mathrm{Gamma}(s, 1)$ and $\mathbf{x} \sim \mathcal{N}(0, \boldsymbol{\Sigma})$ is independent of u, we have*

$$u\boldsymbol{\mu} + \sqrt{u}\mathbf{x} \sim \mathrm{GAL}_p(\boldsymbol{\Sigma}, \boldsymbol{\mu}, s). \tag{1}$$

A proof of this result can be found in Kotz et al. (2001, chap. 6). This representation explains why the multivariate generalized Laplace distribution can also be seen as a multivariate generalization of the *variance-gamma distribution* which is widely used in the field of quantitative finance (Madan et al., 1998). Infinite mixtures similar to (1) are called *variance–mean mixtures* (Barndorff-Nielsen et al., 1982) and are discussed for example by Yu (2017).

Another useful property of the multivariate generalized Laplace distribution is that, under some conditions, it is closed under convolution.

Proposition 2. *Let $s_1, s_2 > 0$, $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma} \in \mathcal{S}_p^+$. If $\mathbf{z}_1 \sim \mathrm{GAL}_p(\boldsymbol{\Sigma}, \boldsymbol{\mu}, s_1)$ and $\mathbf{z}_2 \sim \mathrm{GAL}_p(\boldsymbol{\Sigma}, \boldsymbol{\mu}, s_2)$ are independent random variables, then*

$$\mathbf{z}_1 + \mathbf{z}_2 \sim \mathrm{GAL}_p(\boldsymbol{\Sigma}, \boldsymbol{\mu}, s_1 + s_2). \tag{2}$$