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A test procedure for uniformity on the Stiefel manifold based on projection

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ABSTRACT

This paper proposes a new procedure to test uniformity on the Stiefel manifold. The theoretical analysis of the test procedure, and numerical experiments are conducted to illustrate the usage and the efficiencies through the power under alternative hypotheses.

1. Introduction

Let X be a $p \times r$ ($p \geq r$) random matrix which satisfies $X'X = I_r$, where A' denotes the transpose of the matrix A, and I_d is the $d \times d$ identity matrix. Consider the testing problem of the null hypothesis

$$H_0 : X \text{ are uniformly distributed over } V_r(\mathbb{R}^p), \tag{1}$$

where $V_r(\mathbb{R}^p)$ stands for the *Stiefel manifold* of orthonormal r-frames in the *Euclidean space* $\mathbb{R}^p$.

To address the hypothesis (1), Jupp (2001) reviewed the Rayleigh test and proposed a procedure based on the modified Rayleigh's statistic defined by

$$S^* = \left(1 - \frac{1}{2N} + \frac{1}{2(pr+2)N} S\right) S, \tag{2}$$

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to test uniformity under the matrix Fisher distribution, where $S = Np\,\mathrm{tr}(\bar{X}'\bar{X})$, $\bar{X} = (1/N)\sum_{j=1}^{N} X_j$, and $X_1, \ldots, X_N$ are N independent random copies of X. Furthermore, Arnold and Jupp (2013) provided statistics for tests of uniformity and location of the orthogonal axial frames on the quotient manifold $V_r(\mathbb{R}^p)/\mathbb{Z}_2^r$, where $\mathbb{Z}_2^r = \{(\varepsilon_1, \ldots, \varepsilon_r) : \varepsilon_j = \pm 1\}$, and gave examples of location tests for certain parametric models.

In the $r = 1$ case, the Stiefel manifold $V_1(\mathbb{R}^p)$ is equivalent to the unit sphere $\mathbb{S}^{p-1}$ in $\mathbb{R}^p$, thus replacing X with $\boldsymbol{X}$, a p-dimensional random unit vector, the null hypothesis (1) corresponds to

$$H_0^* : \boldsymbol{X} \text{ are uniformly distributed on } \mathbb{S}^{p-1}. \tag{3}$$

For this hypothesis, for example, Fang et al. (1993) proposed a necessary test for sphericity based on non-parametric goodness of fit Wilcoxon-type statistic for two-sample problem by utilizing the well-known fact that if $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ are independent and uniformly distributed on $\mathbb{S}^{p-1}$, then for each constant $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{S}^{p-1}$, all random variables $\boldsymbol{a}_l'\boldsymbol{X}_j$'s have the same distribution.

On the other hand, Cai et al. (2013) considered the asymptotic behavior of the pairwise angles between vectors which are randomly and uniformly distributed on $\mathbb{S}^{p-1}$, and observed the empirical distributions of the maximum and the minimum of angles to test for spherical symmetry (for characteristics of spherical symmetry, see, e.g., Fang and Zhang, 1990, Sections 5.5 and 5.6). In terms of the power of tests for uniformity on $\mathbb{S}^{p-1}$, Figueiredo (2007) compared some typical procedures, namely, Rayleigh's test, Giné's test and Ajne's test through numerical experiments in some dimensions. These procedures hypothesize the von Mises–Fisher distribution as alternative distribution against the uniform distribution in the sphere.

An interesting approach for testing for uniformity on $\mathbb{S}^{p-1}$ was invented by Cuesta-Albertos et al. (2009). Let $\{\boldsymbol{X}_j\}_{j=1}^N$ be an independent random sample from $\mathbb{S}^{p-1}$. Take another random sample $\{\boldsymbol{U}_l\}_{l=1}^k$ of $\mathbb{S}^{p-1}$, independent from $\boldsymbol{X}_j$'s. Under these conditions, Cuesta-Albertos et al. (2009) introduced random variables $Z_{lj} = \boldsymbol{U}_l'\boldsymbol{X}_j$, the random projections of $\boldsymbol{X}_j$ on $\boldsymbol{U}_l$, to test the null hypothesis H_0^* with employing the *Kolmogorov–Smirnov statistic* defined by

$$D_{N;l} = \sup_z \left| F_{N;l}(z) - F_0(z) \right|, \quad l = 1, \ldots, k, \tag{4}$$

and performed numerical experiments, where $F_{N;l}$ and F_0 are the empirical and the population distribution function of Z_{lj}, respectively (see, e.g., Gibbons and Chakraborti, 2010, Section 4.3).

In this paper, we develop a new test procedure for uniformity on the Stiefel manifold patterned after the methodology devised by Cuesta-Albertos et al. (2009). In Section 2, we prepare for some results concerned with matrix spherical distributions which include the uniform distribution on the Stiefel manifold $V_r(\mathbb{R}^p)$ to provide the probability density function (pdf) and the cumulative distribution function (cdf) of Z_{lj}. On the basis of pdf and cdf of Z_{lj}, we describe the procedure to test the null hypothesis H_0^* in substitution for H_0, and run a simulation study to examine the consistency and the power of our test in Section 3.

2. Test statistic and its distribution

Let X be a $p \times r$ random matrix uniformly distributed on the Stiefel manifold $V_r(\mathbb{R}^p)$ in $\mathbb{R}^p$, that is, it holds that $X'X = I_r$. Then the following proposition is given.

Proposition. *Let a $p \times r$ random matrix X be uniformly distributed over the Stiefel manifold $V_r(\mathbb{R}^p)$, $p \geq r$. For a fixed vector $\boldsymbol{a} \in \mathbb{S}^{r-1}(= V_1(\mathbb{R}^r))$ the p-dimensional random vector $X\boldsymbol{a}$ is uniformly distributed on the unit sphere $\mathbb{S}^{p-1}$ in $\mathbb{R}^p$.*

Proof. According to (i) of Lemma 3.1.3 in Fang and Zhang (1990, p. 95), if a $p \times r$ random matrix X is uniformly distributed over $V_r(\mathbb{R}^p)$, then, fixed $\boldsymbol{a} \in \mathbb{S}^{r-1}$, $X\boldsymbol{a}$ is uniformly distributed over $V_1(\mathbb{R}^p)$, namely, $\mathbb{S}^{p-1}$.

Henceforth if an m-dimensional unit vector $\boldsymbol{U}$ is uniformly distributed on the sphere $\mathbb{S}^{m-1}$ on $\mathbb{R}^m$, we will write $\boldsymbol{U} \sim \mathbb{S}^{m-1}$.

Remark 1. Following (ii) of Lemma 3.1.3 by Fang and Zhang (1990, p. 94), it is obvious that if we take a r-dimensional random unit vector $\boldsymbol{V}$ instead of $\boldsymbol{a}$, independent from X, the assertion in the Proposition holds since $X\boldsymbol{V} \stackrel{d}{=} X\boldsymbol{a}$.

By making use of the result above, it easily follows that if $\{X_j\}_{j=1}^N$ is a uniform random sample drawn from $V_r(\mathbb{R}^p)$, then the N random vectors $\{X_j\boldsymbol{a}\}_{j=1}^N$ are all independent and uniformly distributed on $\mathbb{S}^{p-1}$. Hence, hereafter, we would like to focus on a problem of testing the null hypothesis H_0^* defined by (3). On applying Proposition 1 and Lemma 2 in Iwashita and Klar (2014), we obtain the following result which plays an important part in constructing our testing procedure for H_0^*.

Theorem. *Suppose $\{X_j\}_{j=1}^N$ is a random sample from the uniform distribution over the Stiefel manifold $V_r(\mathbb{R}^p)$ and $\boldsymbol{U}$, independent from X_j's, is a p-dimensional random vector drawn from some continuous distribution with support on $\mathbb{S}^{p-1}$, the unit sphere in $\mathbb{R}^p$. Then, for any r-dimensional unit vector $\boldsymbol{a}$, respective N random variables*

$$Z_j = \boldsymbol{U}'X_j\boldsymbol{a}, \quad j = 1, \ldots, N, \tag{5}$$