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Derivative of intersection local time of independent symmetric stable motions[☆]

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ABSTRACT

Let X and $\tilde{X}$ be two mutually independent symmetric stable motions in $\mathbb{R}^1$ with respective indices α and $\tilde{\alpha}$. We show that the intersection local time $\beta_t(x)$ of X and $\tilde{X}$ is differentiable in the spatial variable if $\alpha + \tilde{\alpha} > 3$, and moreover we have that the p-variation of the derivative $\beta_t'(0)$ is zero when $p > \frac{2\alpha\vee\tilde{\alpha}}{\alpha\vee\tilde{\alpha}+\alpha+\tilde{\alpha}-3}$.

1. Introduction and results

In their study of the intrinsic Brownian local time sheet and stochastic area integrals for Brownian motion, Rogers and Walsh (1991a,b, 1990), were led to analyze the following functional

$$A(x,t) = \int_0^t 1_{[0,\infty)}(x - B_s)ds \tag{1.1}$$

where B_t is a 1-dimensional Brownian motion. They showed that $A(B_t, t)$ is not a semimartingale, and in fact showed that the process

$$A(B_t,t) - \int_0^t \mathscr{L}(B_s,s)dB_s \tag{1.2}$$

has finite non-zero 4/3-variation, where $\mathscr{L}^B(x,t) = \int_0^t \delta(B_s - x)ds$ is the local time of Brownian motion B at x. In 2006, Rosen (2005) developed a new approach to study the process $A(t, B_t)$. The motivation and guiding intuition was as follows. If one lets $h(x) := 1_{[0,\infty)}(x)$, then we have formally

$$\frac{d}{dx}h(x) = \delta(x), \qquad \frac{d^2}{dx^2}h(x) = \delta'(x),$$

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where δ is the Dirac delta function and δ' is the derivative of the δ-function in the sense of Schwartz's distribution. Thus, a formal application of Itô's formula for h yields (see Rosen (2005))

$$A(B_t,t)-\int_0^t \mathscr{L}^B(s,B_s)dB_s = t+\frac{1}{2}\int_0^t\int_0^s \delta'(B_s-B_r)drds. \tag{1.3}$$

However, as Jung and Markowsky (2014) remarks, there is some ambiguity, since if we change the definition of h only slightly to $h(x) := 1_{(0,\infty)}(x)$, change the definition of A to

$$A(t,x)=\int_0^t 1_{(0,\infty)}(x-B_s)ds,$$

and apply Itô's formula in the same manner, we get

$$A(B_t,t)-\int_0^t \mathscr{L}^B(s,B_s)dB_s = \frac{1}{2}\int_0^t\int_0^s \delta'(B_s-B_r)drds.$$

This observation shows that care must be taken, as the two different definitions of $A(t,x)$ agree almost surely. Further details can be found in Jung and Markowsky (2014) and Markowsky (2008). Thus, using the above Itô formula, as a motivation, Rosen (2005) showed the existence of the process

$$\alpha_t'(a) := \int_0^t\int_0^s \delta'(B_s-B_r-a)drds, \tag{1.4}$$

which is called the derivative of self-intersection local time (DSLT) of Brownian motion B. Moreover, Rosen (2005) introduced also some interesting issues, and Jung and Markowsky (2014, 2015) and Yan and Yang (2008); Yan and Yu (2015) extended this to fractional Brownian motion and studied some related questions.

On the other hand, for a stochastic process $\{G(x,t), x\in\mathbb{R}, t\geq 0\}$ satisfying

(1) for each $x\in\mathbb{R}$, $\{G(x,t), t\geq 0\}$ is adapted, and
(2) $\frac{\partial G}{\partial x}(x,t)$ and $\frac{\partial^2 G}{\partial x^2}(x,t)$ are continuous a.s.

Eisenbaum (2000) introduced the following Itô formula (see also Sznitman (1982) and Kunita (1990) for some special cases):

$$G(W_t,t)=G(W_0,0)+\int_0^t G(W_s,s)ds+\int_0^t \frac{\partial G}{\partial x}(W_s,s)dW_s+\frac{1}{2}\int_0^t \frac{\partial^2 G}{\partial x^2}(W_s,s)ds. \tag{1.5}$$

Moreover, to weaken the above hypothesizes (1) and (2), Eisenbaum (2000) constructed the Banach space $\mathscr{H}$ of Borel functions f with the norm

$$\|f\|_{\mathscr{H}} := \left(\int_0^T\int_{\mathbb{R}} f^2(x,s)e^{-\frac{x^2}{2s}}\frac{dxds}{\sqrt{2\pi s}}\right)^{1/2}+\int_0^T\int_{\mathbb{R}}|xf(x,s)|e^{-\frac{x^2}{2s}}\frac{dxds}{s\sqrt{2\pi s}},$$

and introduced the following conditions:

(3) $\frac{\partial G}{\partial x}(x,t)\in\mathscr{H}$ a.s. with bounded variations on compacts;
(4) $\frac{\partial G}{\partial x}(x,t)$ and $\frac{\partial G}{\partial t}(x,t)$ are continuous almost surely, and there exists a Radon measure ν and a locally bounded Borel function H on $\mathbb{R}\times[0,1]$ such that

$$d_x\left(\frac{\partial G}{\partial x}(x,t)\right)=H(x,t)\nu(dx),$$

$t\mapsto H(x,t)$ is continuous on $(0,1)$, $\nu(dx)$-a.s., and for almost every $t\in[0,1]$, $x\mapsto H(x,t)$ is ρ-a.s. continuous, where ρ is the part of ν singular to the Lebesgue measure.

She showed that the integral

$$\int_0^t\int_{\mathbb{R}}\frac{\partial G}{\partial x}(x,s)\mathscr{L}^W(dx,ds)=-\int_0^t\int_{\mathbb{R}} H(x,s)\nu(dx)\mathscr{L}^W(x,ds)$$

exists and the generalized Itô formula

$$G(W_t,t)=G(W_0,0)+\int_0^t\frac{\partial G}{\partial t}(W_s,s)ds+\int_0^t\frac{\partial G}{\partial x}(W_s,s)dW_s-\frac{1}{2}\int_0^t\int_{\mathbb{R}}\frac{\partial G}{\partial x}(x,s)\mathscr{L}^W(dx,ds) \tag{1.6}$$

holds under the conditions (1), (3) and (4). As an attempt to use the Itô formula (1.5) and (1.6), one can naturally consider

$$G(x,t)=A(x,t)=\int_0^t 1_{[0,\infty)}(x-B_s)ds,$$