Statistics and Probability Letters 121 (2017) 45–50

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

On normal variance–mean mixtures

Yaming Yu

Department of Statistics, University of California, Irvine, CA 92697, USA

ARTICLE INFO

Article history:
Received 2 May 2016
Received in revised form 21 July 2016
Accepted 21 July 2016
Available online 7 October 2016

MSC:
60E05
62E15

Keywords:
Generalized inverse Gaussian distribution
Log-convexity
Modified Bessel function
Normal mixture
Poisson mixture
Total positivity

ABSTRACT

We study shape properties of normal variance–mean mixtures, in both the univariate and multivariate cases, and determine conditions for unimodality and log-concavity of the density functions. We also interpret such results in practical terms and discuss discrete analogues.

1. Introduction

A univariate normal variance–mean mixture (Barndorff-Nielsen et al., 1982) is the distribution of

$$Y = \mu + \beta X + \sigma\sqrt{X}Z \tag{1}$$

where X and Z are independent scalar random variables, $Z \sim \mathrm{N}(0, 1)$, X has a density (the mixing density) supported on $(0, \infty)$, and $-\infty < \mu, \beta < \infty$, $\sigma > 0$ are constants. Equivalently, a normal variance–mean mixture is the distribution of a Brownian motion with drift stopped at an independent random time. Normal variance–mean mixtures encompass a large family of distributions commonly used in many applied fields. A prominent example is the generalized hyperbolic (GH) distribution (Barndorff-Nielsen, 1977). This distribution has an unnormalized density of the form ($y \in \mathbb{R}$)

$$\mathrm{gh}(y; \mu, \lambda, \alpha, \beta, \delta) \propto e^{\beta(y-\mu)} \left(\delta^2 + (y-\mu)^2\right)^{(\lambda-1/2)/2} K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2 + (y-\mu)^2}\right), \tag{2}$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind (Abramowitz and Stegun, 1972). The allowable parameter values are $\mu \in \mathbb{R}$ and

$$\begin{aligned} &\delta \ge 0, \quad |\beta| < \alpha, \quad \text{if } \lambda > 0;\\ &\delta > 0, \quad |\beta| < \alpha, \quad \text{if } \lambda = 0;\\ &\delta > 0, \quad |\beta| \le \alpha, \quad \text{if } \lambda < 0. \end{aligned}$$

E-mail address: yamingy@uci.edu.

http://dx.doi.org/10.1016/j.spl.2016.07.024
0167-7152/

The density (2) arises as the density of Y in (1) if we let $\sigma = 1$ and let the density of X be the generalized inverse Gaussian (GIG; Jørgensen, 1982)

$$\mathrm{gig}(x; \lambda, \chi, \psi) = \frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\psi\chi})} x^{\lambda-1} \exp\left[-\frac{1}{2}(\chi x^{-1} + \psi x)\right], \quad x > 0. \tag{3}$$

Parameters other than the common λ are related by $\chi = \delta^2$ and $\psi = \alpha^2 - \beta^2$.

The density (2) includes the Student t, Laplace, hyperbolic, normal inverse Gaussian, and variance gamma densities as special cases. These are important in modeling financial data; see, e.g., Eberlein and Keller (1995), Barndorff-Nielsen (1997), Eberlein (2001), and Chen et al. (2004). The appearance of modified Bessel functions, however, makes them nontrivial for both theoretical analysis and numerical computation. We refer to Protassov (2004) for computational strategies in parameter estimation. There is also a package in R (Breymann and Lüthi, 2010).

This paper is motivated by a basic property of the GH density, namely its unimodality. A density function $f(x)$ on $\mathbb{R}$ is unimodal, if there exists $x_* \in \mathbb{R}$ such that $f(x)$ increases on $(-\infty, x_*]$ and decreases on $[x_*, \infty)$. Unimodality is an inherently interesting and useful property. Chebyshev's bounds for tail probabilities can be sharpened considerably if the distribution is known to be unimodal (Sellke and Sellke, 1997). Shape properties such as unimodality are also important in random variate generation (Devroye, 1986).

All GH densities are unimodal. Although this can be verified analytically in some cases, it is certainly not obvious from the formula (2). The only general proof of unimodality we know is based on two deep results: (i) GH distributions are self-decomposable (Halgreen, 1979; Shanbhag and Sreehari, 1979; Sato, 2001) and (ii) self-decomposable distributions are unimodal (Yamazato, 1978; Steutel and van Harn, 2003). In this paper, we take another approach and obtain a unimodality condition (Theorem 1) for a general normal variance–mean mixture.

We also consider log-concavity and log-convexity. A nonnegative function f supported on an interval $I \subset \mathbb{R}$ is called log-concave if $\log f$ is concave on I. Log-concavity implies unimodality and is often referred to as "strong unimodality" (Dharmadhikari and Joag-dev, 1988). A positive function f on an interval I is log-convex if $\log f$ is convex on I.

Theorem 1. *Suppose X and Y are related by (1) with densities g and f respectively.*

i *If g is unimodal, then so is f.*
ii *If g decreases on $(0, \infty)$, or $\beta = 0$, then the only mode of f is at μ.*
iii *If g is log-concave, then so is f.*
iv *If g is log-convex on $(0, \infty)$, then f is log-convex on each of $(-\infty, \mu)$ and (μ, ∞).*

Theorem 1 has clear practical implications. According to part (i), one must incorporate a multimodal mixing distribution in order for a normal variance–mean mixture model to capture multimodality in the data. This contrasts with other shape properties; skewness, for example, depends on the parameter β and is not entirely inherited from the mixing distribution. On the other hand, log-concave densities have no heavier than exponential tails. Part (iii) therefore indicates that normal variance–mean mixtures inherit their heavy tails from the mixing distributions. That is, one needs to choose a heavy-tailed mixing distribution in (1) in order to model heavy tails in the data.

In the case of the GH distribution, Theorem 1 specializes to Corollary 1. We obtain a short proof of unimodality and a simple criterion for log-concavity for GH densities.

Corollary 1. *Let $f(y)$ denote the* GH *density given by (2).*

i *The density $f(y)$ is unimodal.*
ii *The mode of $f(y)$ is at μ iff either $\beta = 0$ or $\delta = 0,\ 0 < \lambda \le 1$.*
iii *The density $f(y)$ is log-concave iff $\lambda \ge 1$.*
iv *The density $f(y)$ is log-convex on each of $(-\infty, \mu)$ and (μ, ∞) iff $\delta = 0$ and $0 < \lambda \le 1$.*

Theorem 1 is related to the following result for normal mean mixtures (Dharmadhikari and Joag-dev, 1988; Bertin and Theodorescu, 1995). Suppose $Y|X \sim \mathrm{N}(X, \sigma^2)$, X has density g, and Y has marginal density f. Then (i) if g is unimodal then so is f; (ii) if g is log-concave then so is f. These follow from convolution properties of log-concave functions.

We also consider the multivariate case. A p-variate normal variance–mean mixture is the distribution of

$$Y = \mu + \beta X + \sqrt{X} A Z \tag{4}$$

where μ, β are $p \times 1$ vectors, A is a full-rank $p \times p$ matrix, X is a scalar random variable supported on $(0, \infty)$, and Z is a $p \times 1$ standard normal vector independent of X. Owing to properties of the multivariate normal, such mixtures are closed under marginalization and linear transformations. They have the following feature regardless of the shape of the mixing density ($\|\cdot\|$ denotes the Euclidean norm).

Proposition 1. *Suppose $\beta \neq 0$ in (4). Then for each $t \in \mathbb{R}$, the density of Y, denoted by $f(y)$, has ellipsoidal contours on the hyperplane*

$$\mathcal{H}_t \equiv \mu + \beta t + \left\{ z \in \mathbb{R}^p :\ z^\top (AA^\top)^{-1} \beta = 0 \right\}.$$

Specifically, $f(y)$ depends only on $\|A^{-1}(y - \mu - \beta t)\|$ for $y \in \mathcal{H}_t$.