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A note on symmetrization procedures for the laws of large numbers

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ABSTRACT

In this note, general symmetrization procedures for both the weak and strong laws of large numbers concerning the row sums from arrays of rowwise independent Banach space valued random variables are established. Necessary and sufficient conditions are provided for the weak and strong laws of large numbers to hold in terms of corresponding laws of large numbers for a symmetrized version of the array. Several corollaries of the main result are provided.

1. Introduction and the main results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathbf{B}, \|\cdot\|)$ be a real separable Banach space equipped with its Borel σ-algebra $\mathcal{B}$ (= the σ-algebra generated by the class of open subsets of $\mathbf{B}$ determined by $\|\cdot\|$). A $\mathbf{B}$-valued random variable X is defined as a measurable function from $(\Omega, \mathcal{F})$ into $(\mathbf{B}, \mathcal{B})$.

Symmetrization procedures are among the most basic and powerful tools in probability theory, particularly in the study of the limit theorems for sums of random variables. A simple symmetrization procedure for *the strong law of large numbers* (SLLN) was simultaneously and independently stated in Ledoux and Talagrand (1991) and Li (1988). Let $\{Y_n,\ n \geq 1\}$ be a sequence of $\mathbf{B}$-valued random variables and let $\{Y'_n;\ n \geq 1\}$ be an independent copy of $\{Y_n;\ n \geq 1\}$. It follows from either Lemma 7.1 of Ledoux and Talagrand (1991) or Theorem 3 (3.3) of Li (1988) that

$$Y_n \to 0 \text{ almost surely (a.s.) if and only if } Y_n - Y'_n \to 0 \text{ a.s. and } Y_n \to_{\mathbb{P}} 0, \tag{1.1}$$

where "$\to_{\mathbb{P}}$" stands for convergence in probability.

It is natural then to ask whether one can find an analogous result for *the weak law of large numbers* (WLLN). Also, if we only know that $Y_n - Y'_n \to 0$ a.s., can we conclude that

$$Y_n - y_n \to 0 \quad \text{a.s.}$$

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for some sequence $\{y_n;\ n \geq 1\}$ of **B**-valued elements? For $\mathbf{B} = \mathbb{R}$, it follows from the classical weak symmetrization inequalities (see, e.g., Gut, 2013, p. 134) that

$$Y_n - \mu_n \to_{\mathbb{P}} 0 \text{ if and only if } Y_n - Y_n' \to_{\mathbb{P}} 0,$$

where μ_n is a median of the random variable Y_n, $n \geq 1$. Also it follows the classical strong symmetrization inequalities (see, e.g., Gut, 2013, p. 134) that

$$Y_n - \mu_n \to 0 \text{ a.s. if and only if } Y_n - Y_n' \to 0 \text{ a.s.}$$

In this note, general symmetrization procedures for the WLLN and SLLN for the row sums from an array of rowwise independent **B**-valued random variables are established in the following theorem which is the main result. No independence conditions are imposed on the random variables from different rows of the array.

Theorem 1.1. *Let $\{m_n;\ n \geq 1\}$ be a sequence of positive integers. Let $\{X_{i,n};\ 1 \leq i \leq m_n,\ n \geq 1\}$ be an array of **B**-valued random variables such that for each $n \geq 1$, $X_{1,n}, \ldots, X_{m_n,n}$ are independent. Let $\{X'_{i,n};\ 1 \leq i \leq m_n,\ n \geq 1\}$ be an independent copy of $\{X_{i,n};\ 1 \leq i \leq m_n,\ n \geq 1\}$. Write $\hat{X}_{i,n} = X_{i,n} - X'_{i,n},\ 1 \leq i \leq m_n,\ n \geq 1$. Let α be a positive real number such that*

$$\limsup_{n\to\infty} \max_{1\leq i\leq m_n} \mathbb{P}\left(\|X_{i,n}\| > \alpha\right) < \frac{1}{2}. \tag{1.2}$$

Then the following two symmetrization procedures hold:

(i) *Symmetrization procedure for the WLLN:*

$$\sum_{i=1}^{m_n} X_{i,n} - \sum_{i=1}^{m_n} \mathbb{E}\left(X_{i,n} I\{\|X_{i,n}\| \leq 2\alpha\}\right) \to_{\mathbb{P}} 0 \tag{1.3}$$

if and only if

$$\sum_{i=1}^{m_n} \hat{X}_{i,n} \to_{\mathbb{P}} 0. \tag{1.4}$$

(ii) *Symmetrization procedure for the SLLN:*

$$\sum_{i=1}^{m_n} X_{i,n} - \sum_{i=1}^{m_n} \mathbb{E}\left(X_{i,n} I\{\|X_{i,n}\| \leq 2\alpha\}\right) \to 0 \quad \text{a.s.} \tag{1.5}$$

if and only if

$$\sum_{i=1}^{m_n} \hat{X}_{i,n} \to 0 \quad \text{a.s.} \tag{1.6}$$

Theorem 1.1 allows us to obtain laws of large numbers for $\sum_{i=1}^{m_n} X_{i,n},\ n \geq 1$ by restricting consideration to symmetric random variables and by obtaining corresponding laws of large numbers for $\sum_{i=1}^{m_n} \hat{X}_{i,n},\ n \geq 1$. To the best of our knowledge, Theorem 1.1 (especially Theorem 1.1(i)) is new even when $\mathbf{B} = \mathbb{R}$. The following result follows immediately from Theorem 1.1.

Corollary 1.1. *Under the assumptions of Theorem 1.1, we have*

$$\sum_{i=1}^{m_n} X_{i,n} \to 0 \quad \text{a.s. (resp., in probability)}$$

if and only if

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} \mathbb{E}\left(X_{i,n} I\{\|X_{i,n}\| \leq 2\alpha\}\right) = 0 \quad \text{and} \quad \sum_{i=1}^{m_n} \hat{X}_{i,n} \to 0 \quad \text{a.s. (resp., in probability).}$$

For the special case where $m_n = 1,\ n \geq 1$, we have the following result, which answers the two questions posed above in a Banach space setting provided there exists $\alpha > 0$ satisfying (1.7).