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Strong convergence for the Euler–Maruyama approximation of stochastic differential equations with discontinuous coefficients

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ABSTRACT

In this paper we study the strong convergence for the Euler–Maruyama approximation of a class of stochastic differential equations whose both drift and diffusion coefficients are possibly discontinuous.

1. Introduction

Let us consider the one-dimensional stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \quad x_0 \in \mathbb{R},\ t \in [0,T], \tag{1}$$

where $W := (W_t)_{0 \le t \le T}$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions. Since the solution of (1) is rarely analytically tractable, one often approximates $X = (X_t)_{0 \le t \le T}$ by using the Euler–Maruyama (EM) scheme given by

$$X_t^{(n)} = x_0 + \int_0^t b\left(X_{\eta_n(s)}^{(n)}\right) ds + \int_0^t \sigma\left(X_{\eta_n(s)}^{(n)}\right) dW_s, \quad t \in [0,T],$$

where $\eta_n(s) = kT/n =: t_k^{(n)}$ if $s \in [kT/n, (k+1)T/n)$.

It is well-known that if b and σ are Lipschitz continuous, the EM approximation for (1) converges at the strong rate of order $1/2$ (see Kloeden and Platen, 1995). On the other hand, when b and σ are not Lipschitz continuous, the strong rate

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is less known and it has been a subject of extensive study. In the recent articles (Jentzen et al., 2015; Hairer et al., 2015), it has been shown that for every arbitrarily slow convergence speed there exist SDEs with infinitely often differentiable and globally bounded coefficients such that neither the EM approximation nor any approximation method based on finitely many observations of the driving Brownian motion can converge in absolute mean to the solution faster than the given speed of convergence. The approximation for SDEs with possibly discontinuous drift coefficients was first studied in Gyöngy (1998). It is shown that if the drift satisfies the monotonicity condition and the diffusion coefficient is Lipschitz continuous, then the EM scheme converges at the rate of $1/4$ in pathwise senses. In Halidias and Kloeden (2008), the strong convergence of EM scheme is shown for SDEs with discontinuous monotone drift coefficients. If σ is uniformly elliptic and $(\alpha + 1/2)$-Hölder continuous, and b is of locally bounded variation, it has been shown that the strong rate of the EM in L^1-norm is $n^{-\alpha}$ for $\alpha \in (0, 1/2]$ and $(\log n)^{-1}$ for $\alpha = 0$ (see Ngo and Taguchi, 2016, in press). The strong rate of convergence for SDEs whose drift coefficient b is Hölder continuous is studied in Gyöngy and Rásonyi (2011), Menoukeu Pamen and Taguchi (in press) and Ngo and Taguchi (in press). The above mentioned papers contain just a few selected results and a number of further and partially significantly improved approximation results for SDEs with irregular coefficients are available in the literature; see, e.g., Ankirchner et al. (2016), Chan and Stramer (1998), Hashimoto and Tsuchiya (2014), Hutzenthaler et al. (2012), Kohatsu-Higa et al. (2013), Leobacher and Szölgyenyi (2015), Martinez and Talay (2012), Ngo and Taguchi (2015), Yan (2002) and the references there in.

In this paper we are interested in strong approximation of SDEs with discontinuous diffusion coefficients. These SDEs appears in many applied domains such as stochastic control and quantitative finance (see Cherny and Engelbert, 2005; Akahori and Imamura, 2014). For such SDEs, the existence and uniqueness of solution was studied in Nakao (1972), Le Gall (1984) and Cherny and Engelbert (2005); the weak convergence of EM approximation was shown in Yan (2002). To the best of our knowledge, the strong convergence of the EM approximation of SDEs with discontinuous diffusion coefficient has not been considered before in the literature. It is worth noting that the key ingredients to establish the strong rate of convergence of EM approximation for SDEs with discontinuous drift are either the Krylov estimate (see Kohatsu-Higa et al., 2013; Gyöngy and Rásonyi, 2011) or the Gaussian bound estimate for the density of the numerical solution (Lemaire and Menozzi, 2010; Ngo and Taguchi, 2016, in press). However, these estimates seem no longer available for SDEs with discontinuous diffusion coefficients. Therefore in this paper we develop another method, which is based on an argument with local time, to overcome this obstacle.

The remainder of the paper is structured as follows. In the next section we introduce some notations and assumptions for our framework together with the main results. All proofs are deferred to Section 3.

2. Main results

2.1. Notations

Throughout this paper the following notations are used. For any continuous semimartingale Y, we denote $L_t^x(Y)$ the symmetric local time of Y up to time t at the level $x \in \mathbb{R}$ (see Le Gall, 1984). For bounded measurable function f on $\mathbb{R}$, we define $\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|$. We denote by $L^1(\mathbb{R})$ the space of all integrable functions with respect to Lebesgue measure on $\mathbb{R}$ with semi-norm $\|f\|_{L^1(\mathbb{R})} := \int_{\mathbb{R}} |f(x)|dx$. For each $\beta \in (0, 1]$ and $\kappa > 0$, we denote by $H^{\beta,\kappa}$ the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that there exists a measurable subset $S(f)$ of $\mathbb{R}$ satisfying

(i) $\|f\|_\beta := \|f\|_\infty + \sup_{x<y;[x,y]\cap S(f)=\emptyset} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty$; and

(ii) $C_{\beta,\kappa} := \sup_{K\geq 1} \sup_{\varepsilon>0} \frac{\lambda(S(f)^\varepsilon \cap[-K,K])}{K\varepsilon^\kappa} < +\infty$ where λ denotes the Lebesgue measure on $\mathbb{R}$ and $S(f)^\varepsilon$ is the ε-neighbourhood of $S(f)$, i.e., $S(f)^\varepsilon = \{y \in \mathbb{R} : \text{ there exists } x \in S(f) \text{ such that } |x-y| \leq \varepsilon\}$.

Here are some remarks on the class $H^{\beta,\kappa}$.

Remark 2.1. 1. $H^{\beta,\kappa}$ is a vector space on $\mathbb{R}$, i.e., if $a, b \in \mathbb{R}$ and $f, g \in H^{\beta,\kappa}$ then $af + bg \in H^{\beta,\kappa}$.

2. A bounded function f is called piecewise β-Hölder if there exist a positive constant L and a sequence $-\infty = s_0 < s_1 < s_2 < \cdots < s_m < s_{m+1} = \infty$ such that $|f(u) - f(v)| \leq L|u-v|^\beta$ for any u, v satisfying $s_k < u < v < s_{k+1}$. It is easy to verify that such function $f \in H^{\beta,1}$, $S(f) = \{s_1, \ldots, s_m\}$ and $C_{\beta,1} \leq 2m$.

3. The following ζ is a non-trivial example of function of $H^{\beta,\kappa}$ with $\kappa < 1$. For each $\hat{\beta}, \kappa \in (0, 1)$, we denote

$$\zeta(x) = \begin{cases} \dfrac{x-1}{2x-1} & \text{if } x \leq 0, \\ 1 + \dfrac{\log 2}{\log(n+1)} x^{\hat{\beta}} & \text{if } (n+1)^{-1/(1-\kappa)} \leq x < n^{-1/(1-\kappa)} \text{ and } n \in \mathbb{N}, \\ \dfrac{3x+1}{x+1} & \text{if } x \geq 1. \end{cases} \tag{2}$$

It can be shown that ζ is a strictly increasing function with an infinite number of discontinuous points which are cumulative at 0, $\frac{1}{2} < \zeta < 3$, and $\zeta \in H^{\beta,\kappa}$ with $\beta = \frac{1+\hat{\beta}-\kappa}{2-\kappa}$, $S(\zeta) = \{n^{-1/(1-\kappa)}, n = 1, 2, \ldots\}$ and $C_{\beta,\kappa} \leq 3$.