Statistics and Probability Letters 125 (2017) 160–163

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Proper two-sided exits of a Lévy process

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ARTICLE INFO

Article history:
Received 25 November 2015
Accepted 2 February 2017
Available online 15 February 2017

MSC:
primary 60G51
secondary 60J75

Keywords:
Lévy process
Two-sided exit
Support of measure

ABSTRACT

It is proved that the two-sided exits of a Lévy process are proper, i.e. are not a.s. equal to their one-sided counterparts, if and only if said process is not a subordinator or the negative of a subordinator. Furthermore, Lévy processes are characterized, for which the supports of the first exit times from bounded annuli, simultaneously on each of the two events corresponding to exit at the lower and the upper boundary, respectively are unbounded, contain 0, are equal to $[0, \infty)$.

1. Introduction

Two-sided exits of spectrally one-sided Lévy processes have been extensively studied, e.g. Bertoin (1996, Chapter VII) Sato (1999, Section 9.46) and Kyprianou (2006, Section 8.2), to which is added a great number of scientific papers. Less is known in the general case—though integral transforms of many relevant quantities still admit an analytic representation (Kandakov et al., 2005). But the expressions entering these transforms are complicated, and in particular do not lend themselves easily to analysis. The study of the qualitative aspects of the two-sided exit problem, at least for the case of a general Lévy process, thus appears relevant.

Such study is clearly connected with that of the distributional properties of the running supremum $\overline{|X|}$ of the absolute value $|X|$ of a Lévy process X. However – by contrast to those of the supremum process $\overline{X}$ of X itself, e.g. Pecherskii and Rogozin (1969), Rogozin (1966), Doney and Kyprianou (2006) and Chaumont (2013) –, few such properties appear to have been analyzed in general. There are exceptions, e.g. Simon (2001) and Aurzada and Dereich (2009).

In a minor contribution to this area, the purpose of the present paper is to characterize those Lévy processes for which the supports of the 'laws' of the first exit times from bounded annuli, *simultaneously* on each of the two events corresponding to exit at the lower and the upper boundary, respectively are non-empty (Proposition 3), contain 0 (Proposition 4), are unbounded (Proposition 7), are equal to $[0, \infty)$ (Corollary 9). Propositions 5 and 8 give a further analysis of the cases when, respectively, the second and third of the preceding properties fails, but the first does not.

In terms of practical relevance note that Lévy processes are often used to model the risk process of an insurance company (Kyprianou, 2006, Paragraph 1.3.1 & Chapter 7), or their exponentials are used to model the price fluctuations of stocks (Kyprianou, 2006, Paragraph 2.7.3). Thus, for example, it may be useful to know whether or not (in both cases possibly before or after some time, or in each non-degenerate time interval) (i) an insurer with given initial capital will in fact go bankrupt or its capital will exceed some given level, each with a positive probability; or (ii) a perpetual two-sided barrier option will terminate with a positive probability on each of the two boundaries.

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http://dx.doi.org/10.1016/j.spl.2017.02.002
0167-7152/

2. Setting and notation

Throughout we will let X be a Lévy process (Sato, 1999, Section 1) on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P})$ (it is assumed then that X is $\mathbb{F}$-adapted) satisfying the standard assumptions, with diffusion coefficient σ^2, Lévy measure ν and, when $\int 1 \wedge |x|\nu(dx) < \infty$, drift γ_0 (Sato, 1999, Section 8).

Definition 1 (*Two-Sided Exit Times And Their Laws*)**.** Let $\{a, b\} \subset (0, \infty)$.

(i) For a càdlàg path ω mapping $[0, \infty)$ into $\mathbb{R}$, vanishing at zero, we denote by $T_{a,b}(\omega)$ the first entrance time of ω into the set $\mathbb{R} \setminus (-b, a)$ (i.e. the first exit time of ω from $(-b, a)$).

(ii) We introduce the measures $\lambda^+_{a,b}$ and $\lambda^-_{a,b}$ on $\mathcal{B}([0, \infty))$, so that for $A \in \mathcal{B}([0, \infty))$, $\lambda^-_{a,b}(A) \overset{\text{def}}{=} \mathsf{P}(\{T_{a,b}(X) < \infty\} \cap \{X_{T_{a,b}(X)} \leq -b\} \cap \{T_{a,b}(X) \in A\})$ and $\lambda^+_{a,b}(A) \overset{\text{def}}{=} \mathsf{P}(\{T_{a,b}(X) < \infty\} \cap \{X_{T_{a,b}(X)} \geq a\} \cap \{T_{a,b}(X) \in A\})$.

We shall be concerned then with characterizing the pairs (σ^2, ν) and, when $\int 1 \wedge |x|\nu(dx) < \infty$, further the drifts γ_0, under which the measures $\lambda^\pm_{a,b}$ are non-vanishing (on each non-empty interval of the form $[0, M)$, $[m, \infty)$, respectively $[m, M)$/that is contained in $[0, \infty)$/).

Definition 2 (*Auxiliary Notions/Notation*)**.** For a time $S : \Omega \to [0, \infty]$ we will call $(X(S + t) - X(S))_{t\geq 0}$ (defined on $\{S < \infty\}$) the incremental process of X after S. For a measure ρ on a topological space, $\text{supp}(\rho)$ will be its support. Finally, $a \wedge b := \min\{a, b\}$ (when $\{a, b\} \subset [-\infty, +\infty]$): for measurable sets A and B and a measure λ, $\lambda A \wedge \lambda B > 0$ is thus shorthand for "$\lambda(A) > 0$ and $\lambda(B) > 0$".

3. Results

Now the precise statements follow.

Proposition 3. $\lambda^+_{a,b}[0, \infty) \wedge \lambda^-_{a,b}[0, \infty) > 0$ *for some (then all)* $\{a, b\} \subset (0, \infty)$, *if and only if*

> $\sigma^2 > 0$*; or* $\int |x| \wedge 1\nu(dx) = \infty$*; or* ν *charges* $(-\infty, 0)$ *and* $(0, \infty)$ *both; or else* ν *charges only (and does charge)* $(0, \infty)$ *and* $\gamma_0 < 0$, *or* ν *charges only (and does charge)* $(-\infty, 0)$ *and* $\gamma_0 > 0$,

i.e. if and only if neither X *nor* $-X$ *is a subordinator.*

Proof. For the last equivalence see Sato (1999, p. 137, Theorem 21.5). The condition is clearly necessary.

Sufficiency. Let $\{a, b\} \subset (0, \infty)$. The condition implies X is not the zero process, so that $\limsup_\infty X = \infty$ or $\liminf_{-\infty} X = -\infty$ a.s. Sato (1999, p. 255, Proposition 37.10), and so a.s. $T_{a,b}(X) < \infty$. Suppose furthermore *per absurdum*, and then without loss of generality, that a.s. $X_{T_{a,b}(X)} \geq a$. Let $X^{(0)} \overset{\text{def}}{=} X$, $T^{(0)} \overset{\text{def}}{=} T_{a,b}(X)$. Then by the strong Markov property (Sato, 1999, p. 278, Theorem 40.10) of Lévy processes, inductively, we would find that a.s. for all $k \in \mathbb{N}_0$ the incremental process $X^{(k+1)}$ of X after $T^{(k)}$ would satisfy $X^{(k+1)}(T^{(k+1)}) \geq a$, where $T^{(k+1)} \overset{\text{def}}{=} T_{a,b}(X^{(k+1)})$ would be equal in distribution to $T^{(0)}$ and independent of $\mathcal{F}_{T^{(k)}}$. In particular, since by the right-continuity of the sample paths $\mathsf{E}T^{(0)} > 0$ (indeed $T^{(0)} > 0$ a.s.), and since $(T^{(k)})_{k\in\mathbb{N}_0}$ is an i.i.d. sequence, it would follow from the strong law of large numbers, that with probability one X would be $> -b$ at all times. According to Sato (1999, p. 149, Theorem 24.7) this would only be possible if $\sigma^2 = 0$, $\int |x| \wedge 1\nu(dx) < \infty$ with ν charging only $(0, \infty)$. Then according to the assumed condition we would need to have $\gamma_0 < 0$, yielding a contradiction with Sato (1999, p. 151, Corollary 24.8), which necessitates the infimum of the support of X_t being $\gamma_0 t$, for all $t \in [0, \infty)$, in this case. □

In various subcases, this statement can be made more nuanced.

Proposition 4. *The condition that* $\lambda^+_{a,b}[0, M) \wedge \lambda^-_{a,b}[0, M) > 0$ *for all* $M \in (0, \infty]$, *for some (then all)* $\{a, b\} \subset (0, \infty)$, *is equivalent to*

> ν *charges* $(-\infty, 0)$ *and* $(0, \infty)$ *both; or* $\sigma^2 > 0$*; or* $\int 1 \wedge |x|\nu(dx) = \infty$.

Proof. The condition is necessary. For, if $\sigma^2 = 0$, $\int 1 \wedge |x| < \infty$ and, say, ν charges only $(0, \infty)$, then in order that X not have monotone paths, it will need to assume a strictly negative drift, but even then, according to the Lévy–Itô decomposition (Applebaum, 2009, Section 2.4), for given a and b, M can clearly be chosen so small, that by time M, a.s. X can only have left $(-b, a)$ at the upper boundary.

Sufficiency. The argument is similar as in the proof of the preceding proposition, so we forego explicating some of the details. Let $\{a, b\} \subset (0, \infty)$, $M \in (0, \infty)$. Suppose *per absurdum*, and then without loss of generality, that $X_{T_{a,b}(X)} \geq a$ a.s. on $\{T_{a,b}(X) < M\}$. Let $X^{(0)} \overset{\text{def}}{=} X$, $T^{(0)} \overset{\text{def}}{=} T_{a,b}(X)$. By the strong Markov property of Lévy processes, inductively, we find that a.s. for all $k \in \mathbb{N}_0$ the incremental process $X^{(k+1)}$ of X after $T^{(k)}$ satisfies $X^{(k+1)}(T^{(k+1)}) \geq a$ on $\{T^{(k+1)} < M\}$, where $T^{(k+1)} \overset{\text{def}}{=} T_{a,b}(X^{(k+1)})$ is equal in distribution to $T^{(0)}$ and independent of $\mathcal{F}_{T^{(k)}}$. From the strong law of large numbers, it now follows, that with probability one X is $> -b$ on $[0, M)$. But this is only possible if $\sigma^2 = 0$, $\int |x| \wedge 1\nu(dx) < \infty$ with ν charging only $(0, \infty)$, a contradiction. □