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Stochastic accessibility on Grushin-type manifolds

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ABSTRACT

We consider a non-smooth Grushin-type distribution, defined on $\mathbb{R}^n$, whose stochastic perturbation defines the admissible stochastic processes. Our main result is a stochastic accessibility theorem on the corresponding Grushin manifold. More specifically, given two points P and Q, we show how to steer an admissible stochastic processes, starting at P, such that it strikes an arbitrarily small ball centered at Q, asymptotically almost surely.

1. Introduction

Consider an n-dimensional, connected, smooth manifold, denoted by M. A sub-Riemannian structure on M is induced by a given distribution $\mathcal{G}$, which assigns to each point $p \in M$ a k-dimensional subspace $\mathcal{G}_p \subseteq T_pM$, together with a sub-Riemannian metric $g : \mathcal{G} \times \mathcal{G} \to \mathcal{F}(M)$, where $\mathcal{F}(M)$ denotes the ring of smooth functions on M. Usually, the given distribution is non-integrable, has the rank $k < n$, and is spanned locally by a family of k smooth vector fields $\{X_1, X_2, \ldots, X_k\}$. As a general reference for the subject, the reader might consult Bellaïche and Risler (1996), Calin and Chang (2009), Montgomery (2002). One of the features of the sub-Riemannian geometry is the existence of the so-called "*missing directions*" which do not belong to the fixed distribution. The natural curves on sub-Riemannian manifolds are those whose tangent vector fields belong to the fixed distribution. These are called *horizontal curves*.

A very important result in the context of sub-Riemannian geometry is the Chow–Rashevskii Theorem (Chow, 1939). It gives a sufficient condition for the global connectivity, by horizontal curves, to hold. The condition is that the given distribution is bracket-generating (also known as Hörmander's condition Hörmander, 1967), i.e., the vector fields X_i, $i = 1, \ldots, k$, together with their iterated Lie brackets span the whole tangent space T_pM at any point $p \in M$. The problem of connectivity can be also stated in the language of Control Theory as an accessibility problem.

The problem of accessibility in a stochastic setting, which is obtained by perturbing stochastically a given distribution, was raised and motivated by Calin et al. (2014a,b). A stochastic version of the Chow–Rashevskii Theorem has been proved for $\mathbb{R}^2$ endowed with a step 2 Grushin distribution. Their result has been recently generalized to the case when the distribution is given by $\{\partial_{x_1}, x_1^k \partial_{x_2}\}$, $k \in \mathbb{N}^*$ (see Ţurcanu and Udrişte, 2017). For a recent book on probability problems in a geometric framework we refer the reader to Calin and Udrişte (2014). For a detailed study of the geometry induced by the Grushin

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distribution, as well as the analytical properties of the corresponding second order hypoelliptic operator, we refer the reader to Bellaïche and Risler (1996), Calin and Chang (2009), Calin et al. (2005), Chang et al. (2009), Chang and Li (2012) and Ţurcanu (2017).

It is important to notice that in the stochastic setting, the role of admissible (horizontal) curves, is played by the *admissible stochastic processes*. Similarly, the deterministic boundary conditions are reformulated in probabilistic terms. The probability that a stochastic process X_t, starting at a given point P, reaches another fixed point Q is zero. Therefore we ask for the process to reach an arbitrary small ball centered at Q asymptotically almost surely.

In this paper our main goal is to prove the accessibility property by an admissible process driven by a stochastic perturbation of a more general Grushin-type distribution.

At this point, we also mention that throughout this paper no standard summation convention is used and we convene that 0^0 stands for 1.

2. Admissible stochastic processes

Let $k_j \in [0, \infty)$, $j = 1, \dots, n-1$ and consider the vector fields

$$\begin{aligned} X_1 &= \partial_{x_1} \\ X_2 &= |x_1|^{k_1}\partial_{x_2} \\ X_3 &= |x_1|^{k_1}|x_2|^{k_2}\partial_{x_3} \\ &\vdots \\ X_n &= |x_1|^{k_1}|x_2|^{k_2}\cdots|x_{n-1}|^{k_{n-1}}\partial_{x_n}, \end{aligned} \tag{1}$$

defined on $\mathbb{R}^n$. These vector fields generate the non-smooth distribution $\mathcal{G}$ which assigns to each point $x \in \mathbb{R}^n$ the span of the tangent vectors $\{(X_i)_x \mid i = 1, \dots, n\}$.

The distribution $\mathcal{G}$ is smooth and assigns the entire tangent space at each point $x \in \mathbb{R}^n \setminus S$ (*regular points*), where $S := \left(\bigcup_{i=1}^{n-1}\{x_i = 0\}\right)$. On the other hand, the distribution $\mathcal{G}$ is non-smooth, drops its rank, and does not satisfy in general the bracket generating condition (known also as Hörmander's condition Hörmander, 1967) for $x \in S$ (*singular points*).

The vector fields (1) induce a metric space structure which is given by the Carnot–Carathéodory distance

$$d_C(P, Q) = \inf_{c(t)} \int_0^1 \left(\sum_{i=1}^n \frac{\dot{x}^2(t)}{\left(\mu^i(x)\right)^2}\right)^{1/2} dt,$$

where $\mu^i(x) = |x_1|^{k_1}|x_2|^{k_2}\cdots|x_{i-1}|^{k_{i-1}}$, and the infimum is taken over the set of absolutely continuous curves $c : [0, 1] \to \mathbb{R}^n$ such that $c(0) = P$, $c(1) = Q$. The resulting metric space $\mathbb{G}^n = (\mathbb{R}^n, d_C)$ is called *Grushin manifold* (see also Wu (2015)).

It is easy to see that in the deterministic setting, even if the Hörmander's condition is not satisfied in general, the global connectivity still holds.

This means that, given two points P and Q, respectively, we are looking for curves

$$x : [0, t] \to \mathbb{R}^n, \qquad x(s) = (x_1(s), \dots, x_n(s))$$

with $x(0) = P$, $x(t) = Q$, and some controls $u_1(s), \dots, u_n(s)$, such that the tangent vector field writes as

$$\dot{x}(s) = \sum_{i=1}^n u_i(s) X_i\left(x(s)\right). \tag{2}$$

The control functions are supposedly smooth and take values in a bounded and closed set $U \in \mathbb{R}$. Their set is denoted by $\mathcal{U}$ and is called the set of *admissible controls*. The equality (2) writes in coordinate form as

$$\dot{x}_i(s) = u_i(s)\mu^i(x), \quad i = 1, \dots, n,$$

which in turn can be rewritten as Pfaffian system

$$dx_i(s) = u_i(s)\mu^i(x)ds, \quad i = 1, \dots, n. \tag{3}$$

By means of an n-dimensional Wiener process $\left(W_s^1, \dots, W_s^n\right)$, one can add a stochastic effect to the Pfaffian system (3) and thus obtain the SDE (stochastic differential equation) system

$$dx_i(s) = u_i(s)\mu^i(x)ds + \sigma_i dW_s^i, \quad i = 1, \dots, n, \tag{4}$$

where σ_i, $i = 1, \dots, n$ are non-negative real constants controlling the amplitude. It is worth mentioning that the Wiener processes W_s^i, $i = 1, \dots, n$, are supposedly independent.

Stochastic controlled dynamics is usually described (see for instance Øksendal, 2003) by an n-dimensional Itô process

$$x_s = (x_1(s), x_2(s), \dots, x_n(s)),$$