Statistics and Probability Letters 125 (2017) 220–226

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Asymptotic distribution of the conditional-sum-of-squares estimator under moderate deviation from a unit root in MA(1)

Ryota Yabe

Shinshu University, Japan

ARTICLE INFO

Article history:
Received 17 November 2014
Received in revised form 26 January 2017
Accepted 14 February 2017
Available online 24 February 2017

Keywords:
Moving average
Unit root
Moderate deviations

ABSTRACT

We consider the conditional-sum-of-squares estimator (CSSE) for the moderate deviation moving average (MA(1)) process, which has a parameter belonging to a neighborhood of unity with a shrinking radius larger than $O(T^{-1})$ of the near unit root. In this process, we prove consistency and asymptotic normality of the CSSE.

1. Introduction

The unit root literature has often considered a neighborhood of unity with a shrinking radius. For example, near unit roots often appear as local alternatives in unit root tests. Moderate deviation also belongs to the shrinking neighborhood class and has a larger distance from unity than from the near unit root. An asymptotic property of moderate deviation in the AR(1) process has been variously explored to investigate the asymptotic properties in the neighborhood of unity. In a stationary region, Giraitis and Phillips (2006) and Phillips and Magdalinos (2007) showed that an ordinary least squares estimator uniformly has an asymptotically normal distribution. Moreover, its rate of convergence under moderate deviation continuously changes from being stationary to a unit root process depending on the rate of the shrinking radius. These studies showed that both the moderate deviation and the stationary processes share some similar asymptotic properties. Phillips et al. (2010) considered the smoothness of the distributions from stationary to the (near) unit root using the Edgeworth expansion. Giraitis and Phillips (2012) derived the limiting distributions of the sample mean and the auto-covariance to discuss the various asymptotic properties of the stationary, (near) unit root, and moderate deviation processes. See Giraitis and Phillips (2012) for the details and discussion of the asymptotic results.

A (near) unit root MA process that belongs to the local-to-unity class is deeply connected with the KPSS-type test in the AR process. That is, the null hypothesis states that the process is stationary, and the alternative is non-stationary. This testing problem has been studied by several researchers. For example, see Kwiatkowski et al. (1992), and Saikkonen and Luukkonen (1993b,a). In this paper, we consider the moderate deviation MA(1) process, which appears in several empirical studies. It is well known that the ARIMA-based model for macroeconomic data often has the largest MA root close to one, so that the usual asymptotic theory does not work well in finite samples. The MA process with the root close to one has been studied by modeling the MA coefficient as a moderate deviation. For example, Pantula (1991) and Nabeya and Perron (1994) have theoretically shown that the unit root AR process with the moderate deviation MA(1) error term does not follow

E-mail address: ryotayabe@shinshu-u.ac.jp.

http://dx.doi.org/10.1016/j.spl.2017.02.017

the conventional unit root asymptotic theory. Ng and Perron (2001) has proposed the AR unit root test by correcting the information criteria for the special case of moderate deviation.

There are many studies on the estimation theory for a noninvertible process. However, the derivation of the asymptotic distribution of the maximum likelihood estimator (MLE) remains unsolved. This difficulty arises from the asymptotic properties of the MLE. The MLE of the unit root MA(1) process asymptotically equals unity with a positive probability, which is known as the pile-up effect. In other words, the distribution function of the MLE is asymptotically discontinuous at unity, so that the asymptotic distribution does not follow the conventional continuous distribution as normal. This pile-up effect found by Sargan and Bhargava (1983) and Anderson and Takemura (1986) renders the usual limit theory, such as the central limit theorem, inapplicable, which was also observed by Davis and Dunsmuir (1996) in a near unit root process. No study has till now succeeded in deriving the asymptotic distribution of the MLE in a shrinking neighborhood class. However, Yabe (2012) conjectured that an estimator such as the MLE in a moderate deviation process is asymptotically normal as in the invertible case. This paper provides the asymptotic distribution of the conditional-sum-of-squares estimator (CSSE) for the moderate deviation MA(1) process.

The CSSE is known to coincide with the MLE in cases where the initial value of the disturbance is zero. See Harvey (1993) for details on the CSSE in an MA(1) process. We concentrate on the zero-initial-value case, because its asymptotic properties differ from that for the nonzero-initial-value case. We show that the CSSE is consistent and its rate of convergence continuously changes from an invertible to a noninvertible process similar to the AR(1) moderate deviation process. The asymptotic distribution of the CSSE is then normal, even though the process belongs to the shrinking neighborhood class and is same as that in the invertible process. See Harvey (1993) for the results of asymptotic normality of the CSSE in the invertible process.

The rest of this paper is organized as follows. In Section 2, we introduce the model and assumptions. In Section 3, we define the CSSE and derive its asymptotic distribution. Section 4 provides some concluding remarks. All proofs of the theoretical results are presented in the Appendix.

Throughout, we use the following notations: $\to_p$ and $\Rightarrow$ denote convergence in probability and weak convergence, respectively, as the sample size T goes to infinity.

2. Model, assumptions, and CSSE

We assume that the MA(1) data-generating process $\{y_t\}$ is given by:

$$y_t = \epsilon_t - \rho_T \epsilon_{t-1} \quad (t = 1, \ldots, T). \tag{1}$$

We impose the following regularity conditions on $\{\epsilon_t\}$.

Assumption 1. $\{\epsilon_t\}$ is a martingale difference sequence with the variance $\mathrm{E}[\epsilon_t^2|\mathcal{F}_{t-1}] = \sigma^2$, where $\mathcal{F}_{t-1}$ is a natural filtration: $\sigma(\epsilon_{t-1}, \ldots, \ldots, \epsilon_0)$. We also assume that $\{\epsilon_t^2\}$ is uniformly integrable, $\epsilon_0 = 0$ and $y_t = 0$ almost surely (a.s.) for $t \le 0$.

The initial value condition for ϵ_t is crucial, because the nonzero initial value causes different asymptotic properties in the (near) unit root MA(1) model. See Tanaka (1996) for details on the effect of the initial-value assumption on the MLE and the test statistic. Moreover, as noted by Harvey (1993), the CSSE does not coincide with the MLE in the nonzero-initial-value case. Although the analysis of different initial-value assumptions is interesting, further study in this direction is beyond the scope of the present work.

The moderate deviation MA root can be written as $\rho_T = 1 - c/k_T$, where k_T is a deterministic sequence which diverges to infinity and $c > 0$, and is assumed to satisfy the following condition:

Assumption 2. The coefficient ρ_T depending on T satisfies $\rho_T \to 1$, $T(1-\rho_T) = cT/k_T \to \infty$ and $\rho_T \in (-1, 1)$.

From the definition of the coefficient, note that $\{y_t\}$ is invertible in a finite sample, and its coefficient converges to unity. Although $\{y_t\}$ is a triangular array $\{y_{t,T}\}$, we omit the index T to simplify the notation.

We introduce the CSSE, which is defined as the solution of the following minimization problem:

$$\hat{\rho} = \arg \min_{\rho \in [-1,1]} Q_T(\rho), \tag{2}$$

where

$$Q_T(\rho) = \sum_{t=1}^{T} \left(\sum_{l=0}^{t-1} \rho^l y_{t-l} \right)^2 .$$

$Q_T(\rho)$ is equivalent to the sum of squares of the disturbance terms at $\rho = \rho_T$. The solution of this minimizing problem is restricted to the region $[-1, 1]$ to satisfy identification. Note that in the zero-initial-value case, the objective function $Q_T(\rho)$ is a monotonic transformation of the concentrated likelihood function, and so the CSSE coincides with the MLE even in finite samples. However, in the nonzero-initial-value case, the CSSE is not equal to the MLE. See Harvey (1993) and Box et al. (2013) for details about the properties of CSSE in an invertible process.