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Large deviations for the Ornstein–Uhlenbeck process without tears

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ABSTRACT

Our goal is to establish large deviations for the maximum likelihood estimator of the drift parameter of the Ornstein–Uhlenbeck process without tears. We propose a new strategy to establish large deviation results which allows us, via a suitable transformation, to circumvent the classical difficulty of non-steepness. Our approach holds in the stable case where the process is positive recurrent as well as in the unstable and explosive cases where the process is respectively null recurrent and transient. It can also be successfully implemented for more complex diffusion processes.

1. Introduction

Consider the Ornstein–Uhlenbeck process observed over the time interval $[0, T]$

$$dX_t = \theta X_t dt + dB_t \tag{1.1}$$

where (B_t) is a standard Brownian motion and the drift θ is an unknown real parameter. For the sake of simplicity, we assume that the initial state $X_0 = 0$. The process is said to be stable if $\theta < 0$, unstable if $\theta = 0$, and explosive if $\theta > 0$. The maximum likelihood estimator of θ is given by

$$\widehat{\theta}_T = \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \frac{X_T^2 - T}{2\int_0^T X_t^2 dt}. \tag{1.2}$$

It is well-known that in the stable, unstable, and explosive cases

$$\lim_{T\to\infty} \widehat{\theta}_T = \theta \quad \text{a.s.}$$

The purpose of this paper is to establish large deviation principles (LDP) for $(\widehat{\theta}_T)$ via fairly easy to handle arguments. In the stable case, Florens-Landais and Pham (1999) proved an LDP for the score function defined, for all $c \in \mathbb{R}$, by

$$\int_0^T X_t dX_t - c\int_0^T X_t^2 dt.$$

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Then, they were able to deduce, by contraction, the LDP for $(\widehat{\theta}_T)$. However, one can realize in Lemma 4.3 of Florens-Landais and Pham (1999) that the normalized cumulant generating function of the score function is quite complicated to compute. Moreover, its LDP relies on a sophisticated time varying change of probability.

In the unstable and explosive cases (Bercu et al., 2012), the strategy for proving an LDP for $(\widehat{\theta}_T)$ is also far from being obvious. As a matter of fact, on can observe in Lemma 2.1 of Bercu et al. (2012) that the normalized cumulant generating function is also very complicated to evaluate. Moreover, as the limiting cumulant generating function is not steep, it is also necessary to make use of a sophisticated time varying change of probability.

Our approach is totally different. It will allows us, via a suitable transformation, to circumvent the classical difficulty of non-steepness. The starting point is to establish, thanks to Gärtner–Ellis's theorem Dembo and Zeitouni (1998), an LDP for the couple

$$V_T = \Big(\frac{X_T}{\sqrt{T}}, \frac{S_T}{T}\Big) \tag{1.3}$$

where the energy S_T is given by

$$S_T = \int_0^T X_t^2 dt.$$

Then, we will obtain the LDP for $(\widehat{\theta}_T)$ by a direct use of the contraction principle. We refer the reader to Bercu and Richou (2015) where our approach was already implemented for the stable Ornstein–Uhlenbeck process with shift. We also wish to stress that our strategy could be successfully extended to more complex diffusions such as the Pearson diffusion (Forman and Sorensen, 2008)

$$dX_t = (a + bX_t)dt + \sqrt{\alpha X_t^2 + \beta X_t + \gamma}\, dB_t$$

where α, β and γ are chosen such that the square root is well defined for any X_t in the state space. In particular, our approach could be extended to the Jacobi diffusion (Alfonsi, 2015; Demni and Zani, 2009; Zhao and Gao, 2010)

$$dX_t = (a + bX_t)dt + 2\sqrt{1 - X_t^2}\, dB_t$$

where $a \geq 4 + b$ and $a + b \leq -4$, as well as to the Wright–Fisher diffusion (Alfonsi, 2015)

$$dX_t = (a + bX_t)dt + 2\sqrt{X_t(1 - X_t)}\, dB_t$$

where $a \geq 2$ and $a + b \leq -2$. Furthermore, LDP for the estimators of the unknown parameters of the Cox–Ingersoll–Ross diffusion

$$dX_t = (a + bX_t)dt + 2\sqrt{X_t} dB_t$$

where $a > 2$ and $b < 0$ can be found in Du Roy de Chaumaray (2016) and Zani (2002). It still remains to investigate the explosive case $b > 0$.

The paper is organized as follows. In Section 2, we establish an LDP for the couple given by (1.3) and we deduce by contraction the LDP for $(\widehat{\theta}_T)$ in the stable, unstable, and explosive cases. Standard tools for proving LDP such as the Gärtner–Ellis theorem and the contraction principle are recalled in Appendix A, while all technical proofs of Section 2 are postponed to Appendix B.

2. Large deviations

The usual notions of full and weak LDP are as follows.

Definition 2.1. A sequence of random vectors (V_T) of $\mathbb{R}^d$ satisfies an LDP with speed T and rate function I if I is a lower semicontinuous function from $\mathbb{R}^d$ to $[0, +\infty]$ such that,

(i) Upper bound: For any closed set $F \subset \mathbb{R}^d$,

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{P}\big(V_T \in F\big) \leq -\inf_{x\in F} I(x). \tag{2.1}$$

(ii) Lower bound: For any open set $G \subset \mathbb{R}^d$,

$$-\inf_{x\in G} I(x) \leq \liminf_{n\to\infty} \frac{1}{T} \log \mathbb{P}\big(V_T \in G\big). \tag{2.2}$$

Moreover, I is said to be a good rate function if its level sets are compact.

Definition 2.2. A sequence of random vectors (V_T) of $\mathbb{R}^d$ satisfies a weak LDP with speed T and rate function I if I is a lower semicontinuous function from $\mathbb{R}^d$ to $[0, +\infty]$ such that the upper bound (2.1) holds for any compact set, while the lower bound (2.2) is true for any open set.

It is well-known that if (V_T) is exponentially tight and satisfies a weak LDP, then I is a good rate function and the full LDP holds for (V_T), see Lemma 1.2.18 of Dembo and Zeitouni (1998).