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# The Tracy–Widom distribution is not infinitely divisible

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ABSTRACT

The classical infinite divisibility of distributions related to eigenvalues of some random matrix ensembles is investigated. It is proved that the $\beta$-Tracy–Widom distribution, which is the limiting distribution of the largest eigenvalue of a $\beta$-Hermite ensemble, is not infinitely divisible. Furthermore, for each fixed $N \geq 2$ it is proved that the largest eigenvalue of a GOE/GUE random matrix is not infinitely divisible.



## 1. Introduction

Random matrix theory is an important field in probability, statistics and physics. One of the aims of random matrix theory is to derive limiting laws for the eigenvalues of ensembles of large random matrices. In this sense this note will focus on the study of the behavior of the eigenvalues of two types of matrix ensembles: the invariant Hermite ensembles and the tridiagonal $\beta$-Hermite ensembles.

The invariant Hermite ensembles consist of the Gaussian orthogonal, unitary, or symplectic ensembles, G(O/U/S)E, which are ensembles of $N \times N$ real symmetric, complex Hermitian or Hermitian real quaternion matrices, $H$, respectively, whose matrix elements are independently distributed random Gaussian variables with probability density function (PDF) proportional, modulo symmetries, to

$$\exp\left(-\frac{\beta}{4}\mathrm{tr}H^2\right).$$

Here, $\beta = 1, 2$ or $4$ is used for the G(O/U/S)E ensembles, respectively. The joint PDF of their ordered eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ is given by

$$k_{N,\beta} \prod_{1\leq i<j\leq N} \left|\lambda_i - \lambda_j\right|^{\beta} \exp\left(-\frac{\beta}{4}\sum_{i=1}^{N}\lambda_i^2\right), \tag{1}$$

where $k_{N,\beta}$ is a non-negative constant and for $\beta = 1, 2$ or $4$, it can be computed by Selberg's Integral Formula (see Anderson et al., 2010, Theorem 2.5.8). The PDF (1) exhibits strong dependence of the eigenvalues of the G(O/U/S)E ensembles. For more details related to these ensembles see Mehta (2004), Deift and Gioev (2009), Forrester (2010) and Anderson et al. (2010, Sections 2.5 and 4.1). The law (1) has a physical meaning since it describes a one-dimensional Coulomb gas at inverse temperature $\beta$, Forrester (2010, Section 1.4).

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Each member of the G(O/U/S)E ensembles leads, by the Householder reduction, to a symmetric tridiagonal matrix $(H_N^\beta)_{N\geq 1}$ of the form

$$H_N^\beta := \frac{1}{\sqrt{\beta}} \begin{bmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{bmatrix}, \tag{2}$$

where $\chi_t$ is the $\chi$-distribution with $t$ degrees of freedom, whose probability density function is given by $f_t(x) = 2^{1-t/2}x^{t-1} e^{-x^2/2}/\Gamma(t/2)$. Here, $\Gamma(\alpha) = \int_0^\infty v^{\alpha-1}e^{-v}dv$ is Euler's Gamma function. The matrix (2) has the important characteristic that all entries in the upper triangular part are independent. Trotter (1984) applied the Householder reduction to the symmetric case ($\beta = 1$) while for the unitary and symplectic cases ($\beta = 2, 4$) that reduction was applied by Dumitriu and Edelman (2002), who also consider the ensemble (2) for general $\beta > 0$ proving that in this case the PDF of the ordered eigenvalues of $H_N^\beta$ is still the PDF (1), see Akemann et al. (2011, Chapter 20) and Anderson et al. (2010, Section 4.1). This matrix model will be referred to as the $\beta$-Hermite ensemble.

The classical Tracy–Widom distribution is defined as the limit distribution of the largest eigenvalue of a G(O/U/S)E random matrix ensemble. It is important due to its applications in probability, combinatorics, multivariate statistics, and physics, among other applications. Tracy and Widom (2002, 2009) have written concise reviews of the situations where their distribution appears.

The $\beta$-Tracy–Widom distribution is defined as the limiting distribution of the largest eigenvalue of a $\beta$-Hermite ensemble. In the case $\beta = 1, 2, 4$ the $\beta$-Tracy–Widom distribution coincides with the classical Tracy–Widom, Ramírez et al. (2011).

The main purpose of this paper is to determine the infinite divisibility of the classical Tracy–Widom and $\beta$-Tracy–Widom distributions as well as the infinite divisibility of the largest eigenvalue of the finite dimensional random matrix of GOE and GUE ensembles.

Recall that a random variable $X$ is said to be *infinitely divisible* if for each $n \geq 1$, there exist independent random variables $X_1, \ldots, X_n$ identically distributed such that $X$ is equal in distribution to $X_1 + \cdots + X_n$. This is an important property from the theoretical and applied point of view, since for any infinitely divisible distribution there is an associated Lévy process; Sato (2013) and Rocha-Arteaga and Sato (2003). These jump processes have been recently used for modeling purposes in a broad variety of different fields, including finance, insurance, and physics, among others; see Barndorff-Nielsen et al. (2001), Cont and Tankov (2003) and Podolskij et al. (2016) and for a physicist's point of view, see Paul and Baschnagel (2013). Other applications concern deconvolution problems in mathematical physics, Carasso (1992).

This note is structured as follows: in Section 2 there are presented preliminary results on the tail behavior of the classical and generalized $\beta$-Tracy–Widom distribution, useful to analyze infinite divisibility. The non-infinite divisibility of the classical and generalized $\beta$-Tracy–Widom distribution is proved in Section 3. In Section 4 it is shown that for each $N \geq 2$, the largest eigenvalue of a G(O/U)E ensemble is not infinitely divisible. Lastly, in Section 5 there are presented more new results and some open problems.

## 2. Tracy–Widom distributions

### 2.1. Classical Tracy–Widom distribution

It is well known that the only possible limit distributions for the maximum of independent random variables are the Gumbel, Fréchet and Weibull distributions. To classify the limit laws for the maximum of a large number of non-independent random variables is still open problem. A possible strategy is to deal with particular models of non-independent random variables.

The eigenvalues of random matrices provide a good example of such non-independent random variables. For the Gaussian ensembles, i.e., for $N \times N$ matrices with independent Gaussian entries, the joint density function of their eigenvalues, $\lambda_1 \leq \cdots \leq \lambda_N$ is given by (1), and because the Vandermonde determinant $\prod_{1\leq i<j\leq N} \left|\lambda_i - \lambda_j\right|^\beta$ they are strongly dependent. Due to the non-independence of random variables with density function (1) it follows that the limit distribution of $\lambda_{\max} = \lambda_N$ is not a usual extreme distribution. The distribution of $\lambda_{\max}$ converges in the limit $N \to \infty$ to the Tracy–Widom laws.

Tracy and Widom (1994, 1996) proved that the following limit, which is denoted by $F_\beta$, exists

$$F_\beta(x) := \lim_{N\to\infty} P\left[N^{1/6}\left(\lambda_{\max} - 2\sqrt{N}\right) \leq x\right], \quad \beta = 1, 2, 4$$

and in this case

$$F_2(x) = \exp\left(-\int_x^\infty (s-x)\left[q(s)\right]^2 ds\right),$$