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A reverse Gaussian correlation inequality by adding cones[☆]

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ABSTRACT

Let γ denote any centered Gaussian measure on $\mathbb{R}^d$. It is proved that for any closed convex sets A and B in $\mathbb{R}^d$, and any closed convex cones C and D in $\mathbb{R}^d$, if $D \supseteq C^\circ$, where C° is the polar cone of C, then

$$\gamma((A+C)\cap(B+D)) \le \gamma(A+C)\cdot\gamma(B+D),$$

and

$$\gamma((A+C)\cap(B-D)) \ge \gamma(A+C)\cdot\gamma(B-D).$$

As an application, this new inequality is used to bound the asymptotic posterior distributions of likelihood ratio statistics for convex cones.

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1. Introduction

There are many inequalities about Gaussian measures over convex sets due to the diverse applications to probability, statistics, econometrics, geometry, quantum physics and other areas. Many such inequalities can be found in some books (e.g., Bogachev, 1998; Ledoux and Talagrand, 1991; Gine and Nickl, 2015) and review papers (e.g., Latala, 2002; Li and Shao, 2001).

Let γ denote any centered Gaussian measure on $\mathbb{R}^d$. The well known Gaussian correlation inequality for *symmetric* convex sets that was conjectured more than 40 years ago states that if K and L are two closed symmetric convex sets in $\mathbb{R}^d$, then

$$\gamma(K\cap L) \ge \gamma(K)\cdot\gamma(L).$$

Earlier partial proofs of this conjecture and applications to frequentist confidence sets can be found in Borell (1981), Schechtman et al. (1998), Khatri (1967), Sidak (1967) and Latala (2002) among others. This conjecture has been recently proved in full generality by Royen (2014); see Latala and Matlak (2015) for an easy-to-follow presentation.

In this note we first establish a reverse Gaussian correlation inequality by adding convex cones. We then provide a simple application to bound the asymptotic posterior distributions of likelihood ratio statistics for convex cones.

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2. A reverse Gaussian correlation inequality by adding cones

In this and the next sections, we use some well-known facts about closed convex cones that are presented in any textbooks on convex analysis, see Hiriart-Urruty and Lemarechal (2001) for example. All the closed convex cones we consider start at zero. For any convex cone $C \in \mathbb{R}^d$ its polar cone is defined as $C^\circ := \left\{x \in \mathbb{R}^d \mid \langle y, x\rangle \le 0,\ \text{for all } y \in C\right\}$. We first establish the following inequality:

Theorem 1. *For any closed convex sets A and B in $\mathbb{R}^d$, and any closed convex cones C and D in $\mathbb{R}^d$, if $D \supseteq C^\circ$, where C° is the polar cone of C, then*

$$\gamma((A+C)\cap(B+D)) \le \gamma(A+C)\cdot\gamma(B+D),$$

and

$$\gamma((A+C)\cap(B-D)) \ge \gamma(A+C)\cdot\gamma(B-D),$$

where γ is any centered Gaussian measure on $\mathbb{R}^d$.

2.1. Proof of Theorem 1

By otherwise replacing A and B by the convex sets $\Sigma^{-1/2}A$ and $\Sigma^{-1/2}B$, and replacing C and D by the convex cones $\Sigma^{-1/2}C$ and $\Sigma^{-1/2}D$, where Σ is a $d \times d$ positive definite matrix, we can assume that γ is the standard Gaussian measure on $\mathbb{R}^d$. Also, by otherwise using approximation, we can assume that the closed convex sets A and B have finitely many extreme points, and that C and D are finitely generated closed convex cones. Suppose that

$$A = \operatorname{conv}\{a_1,\ldots,a_m\}, \qquad B = \operatorname{conv}\{b_1,\ldots,b_n\},$$
$$C = \operatorname{cone}\{c_1,\ldots,c_r\}, \qquad D = \operatorname{cone}\{d_1,\ldots,d_s\},$$

where $c_1,\ldots,c_r,d_1,\ldots,d_s$ are unit vectors. By Minkowski–Weyl Theorem, C° is also finitely generated. So we assume

$$C^\circ = \operatorname{cone}\{c_1^\circ,\ldots,c_t^\circ\},$$

where $c_1^\circ,\ldots,c_t^\circ$ are unit vectors. Because C is a closed convex cone, we have $(C^\circ)^\circ = C$. Thus, we have

$$C = \left\{x \in \mathbb{R}^d \mid \left\langle c_j^\circ, x\right\rangle \le 0,\ 1 \le j \le t\right\}, \tag{1}$$
$$C^\circ = \left\{x \in \mathbb{R}^d \mid \langle c_i, x\rangle \le 0,\ 1 \le i \le r\right\}. \tag{2}$$

Since

$$A + C = \operatorname{conv}\{a_1,\ldots,a_m\} + \operatorname{cone}\{c_1,\ldots,c_r\}$$
$$= \left\{x \in \mathbb{R}^d \mid x = \sum_{j=1}^m \alpha_j a_j + \sum_{i=1}^r \gamma_i c_i : \alpha_1 \ge 0,\ldots,\alpha_m \ge 0, \sum_{j=1}^m \alpha_j = 1, \gamma_1 \ge 0,\ldots,\gamma_r \ge 0\right\},$$

if we let

$$P := \left\{(x,z) \in \mathbb{R}^{d+1} \mid x = \sum_{j=1}^m \alpha_j a_j + \sum_{i=1}^r \gamma_i c_i, z = \sum_{j=1}^m \alpha_j : \alpha_1 \ge 0,\ldots,\alpha_m \ge 0, \gamma_1 \ge 0,\ldots,\gamma_r \ge 0\right\}$$
$$= \operatorname{cone}\{(a_1,1),\ldots,(a_m,1),(c_1,0),\ldots,(c_r,0)\},$$

then,

$$A + C = \left\{x \in \mathbb{R}^d \mid (x,1) \in P\right\}. \tag{3}$$

By Minkowski–Weyl theorem, P° is finitely generated. So, we can assume

$$P^\circ = \operatorname{cone}\{(w_1,\lambda_1),(w_2,\lambda_2),\ldots,(w_k,\lambda_k)\},$$

where $w_1,\ldots,w_k$ are unit vectors in $\mathbb{R}^d$, and $\lambda_1,\ldots,\lambda_k$ are real numbers. Since the vectors $w_1,\ldots,w_k$ must satisfy

$$\left\langle (c_i,0),(w_j,\lambda_j)\right\rangle \le 0, \quad 1 \le i \le r,\ 1 \le j \le k,$$

we have

$$\left\langle c_i, w_j\right\rangle \le 0, \quad 1 \le i \le r,\ 1 \le j \le k.$$

Thus, by the definition of C°, we have $w_j \in C^\circ$ for all $1 \le j \le k$. Since $D \supseteq C^\circ$, we have $w_j \in D$ for $1 \le j \le k$.

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