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A simple nonparametric method to estimate the expected time to cross a threshold

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HIGHLIGHTS

- A nonparametric method to estimate the expected time to cross a threshold is proposed.
- Only two assumptions are considered: Markovian property and stationarity.
- The estimator is applied to real exchange rates.

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ABSTRACT

We propose a simple but effective nonparametric method to estimate the expected time to cross a threshold using only two assumptions: Markovian property and stationarity. We provide an empirical application with real exchange rates to illustrate the method.

1. Introduction

First hitting time and especially the expected time to cross some thresholds are fundamental concepts in stochastic analysis, and yet they have received little attention in economics. One of the reasons is probably the difficulty in obtaining a simple procedure to calculate, for example, the expected time (ET) to reach a threshold. In fact, analytical results on first hitting time problems are mostly based on stochastic processes of diffusion type or Markov chains where explicit analytical expressions are usually available.

First hitting times are often used in mathematical finance, biology and other life sciences, where the use of Markov chains and stochastic differential equations is more common, to study, for example, time to extinction or default (in finance). Nonetheless, ET may also be a very useful tool in economics as a way to discuss topics such as the speed of mean-reversion, the time to equilibrium, the time to recovery or recession, etc. A specific example is provided in Section 3.

In this paper we propose a new estimator in a complete nonparametric framework to estimate the expected time of a process to cross a threshold using only two assumptions: Markovian property and stationarity. We also derived the standard errors of the estimator. Section 2 presents the estimator and Section 3 illustrates it with an empirical application to the real exchange rates.

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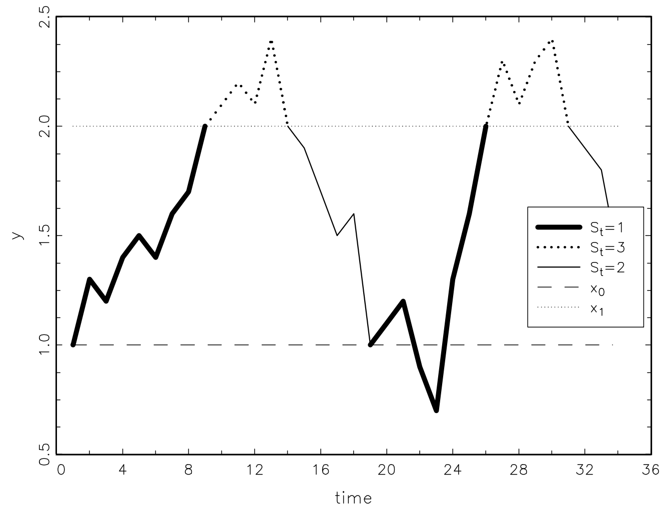

Fig. 1. Illustrating map (1), where $x_0 = 1, x_1 = 2$. Thick line: $S_t = 1$; thin line: $S_t = 2$; dot line: $S_t = 3$.

2. The nonparametric estimator

Let y be a discrete-time process with state space $\mathbb{R}$. We assume that: (A1) y is a Markov process of order r, and (A2) y is a strictly stationary process. From A2 it follows that y is positive Harris recurrent in the following sense (see Meyn and Tweedie, 2012, chap. 9). Let A be a measurable set of the range D of the process of interest, and define the first hitting time of A as $T_A = \inf\{t > 0 : y_t \in A\}$. Therefore there is a σ-finite measure $m(dy)$ such that $m(A) > 0$ implies $\mathrm{E}(T_A | X_0 = a) < \infty$ for every $a \in D \backslash \bar{A}$ where $\bar{A}$ is the closure of the set A. Under assumption A2, it can be proved that the process starting from a level a not belonging to the generic set A, the process y visits A an infinite number of times as $t \to \infty$, almost surely (see Meyn and Tweedie, 2012, chap. 9). This property is of course crucial for (pointwise) identification, as we will see later.

We consider the hitting time $T := T_{x_1} = \min\{t > 0 : y_t \geq x_1\}$ and suppose that the process starts at value $x_0 < x_1$. The case $x_0 > x_1$ with $T_{x_1} = \min\{t > 0 : y_t \leq x_1\}$ is almost analogous. A brief remark on this case will be made later on. The distribution of T is usually difficult to deduce from general non-linear processes. However, there is a simple nonparametric method to estimate these quantities. Set $S_0 = 1$ if $y_0 = x_0$ (note that the process starts at $y_0 = x_0$). Now define the following transformation for $k \geq 0$

$$S_t = \begin{cases} 1 & \text{if } y_t < x_1, y_{t-1} < x_1, \ldots, y_{t-k+1} < x_1, y_{t-k} \leq x_0 \\ 2 & \text{if } x_0 < y_t \leq x_1, x_0 < y_{t-1} \leq x_1, \ldots, x_0 < y_{t-k+1} \leq x_1, y_{t-k} \geq x_1 \\ 3 & \text{otherwise.} \end{cases} \tag{1}$$

Fig. 1 illustrates the map (1) for a hypothetical trajectory of y.

The probabilities of T can be obtained from process S_t. In fact

$$\begin{aligned} P(T = 1) &= P(S_1 > 1 | S_0 = 1) = 1 - P(S_1 = 1 | S_0 = 1) \\ P(T = 2) &= P(S_2 > 1, S_1 = 1 | S_0 = 1) \\ &= P(S_2 > 1 | S_1 = 1, S_0 = 1) P(S_1 = 1 | S_0 = 1) \\ &= (1 - P(S_2 = 1 | S_1 = 1, S_0 = 1)) P(S_1 = 1 | S_0 = 1) \end{aligned}$$

and in general

$$P(T = t) = (1 - p_t) \prod_{i=1}^{t-1} p_i = (1 - p_t) p_{t-1} p_{t-2} \ldots . p_1$$

where $p_t = P(S_t = 1 | S_{t-1} = 1, S_{t-2} = 1, \ldots, S_0 = 1)$. Our strategy is to treat S_t as a Markov chain with state space $\{1, 2, 3\}$ from which we then estimate the relevant parameters. The following result supports our approach.

Proposition 1. *Suppose that y is a rth order Markov process. Then S is a rth order Markov chain.*

Proof. We illustrate the rth order Markov property only for the probability $P(S_t = 1 | S_{t-1} = 1, \ldots, S_0 = 1)$. The other cases are similar. The event $\{S_{t-1} = 1, S_{t-1} = 1, \ldots, S_0 = 1\}$ represents $\{y_{t-1} < x_1, y_{t-2} < x_1, \ldots, y_1 < x_1, y_0 \leq x_0\}$, therefore the probability of $S_t = 1$ given $A = \{S_{t-1} = 1, S_{t-1} = 1, \ldots, S_0 = 1\}$ is equivalent to the probability of $y_t < x_1$