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A refined version of the integro-local Stone theorem

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A B S T R A C T

Let $X, X_1, X_2, \ldots$ be a sequence of non-lattice i.i.d. random variables with $\mathbf{E}\,X = 0$, $\mathbf{E}\,X = 1$, and let $S_n := X_1 + \cdots + X_n$, $n \ge 1$. We refine Stone's integro-local theorem by deriving the first term in the asymptotic expansion, as $n \to \infty$, for the probability $\mathbf{P}(S_n \in [x, x + \Delta))$, $x \in \mathbb{R}$, $\Delta > 0$, and establishing uniform in x and Δ bounds for the remainder term, under the assumption that the distribution of X satisfies Cramér's strong non-lattice condition and $\mathbf{E}\,|X|^r < \infty$ for some $r \ge 3$.

1. Introduction and the main result

In the present note, we establish a refinement of the following remarkable integro-local version of the central limit theorem due to C. Stone (Stone, 1965, 1967). Let $X, X_1, X_2, \ldots$ be a sequence of non-lattice independent identically distributed (i.i.d.) random variables (r.v.'s) following a common distribution F such that $\mathbf{E}\,X = 0$, $\mathbf{E}\,X^2 = 1$, and let $S_n := X_1 + \cdots + X_n$, $n \ge 1$. For $x \in \mathbb{R}$ and $\Delta > 0$, set

$$\Delta[x) := [x, x + \Delta).$$

Then, as $n \to \infty$,

$$\frac{1}{\Delta}\mathbf{P}\big(S_n \in \Delta[x)\big) = n^{-1/2}\phi(xn^{-1/2}) + o(n^{-1/2}), \qquad (1)$$

where $\phi(t) := (2\pi)^{-1/2}e^{-t^2/2}$ is the standard normal density and the remainder term is uniform in $x \in \mathbb{R}$ and in $\Delta \in [\Delta_0, \Delta_1]$ for any fixed $0 < \Delta_0 < \Delta_1 < \infty$ (in fact, C. Stone established more general versions of the above result, including convergence to stable laws, the multivariate case and large deviations).

It is quite appropriate to call relations of the form (1) the *integro-local theorems*, to distinguish them from the integral theorems (which refer to approximating probabilities of the form $\mathbf{P}(S_n < x)$, $x \in \mathbb{R}$) and the local ones (which deal with

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approximating the densities of S_n in the "smooth" case; note that in the arithmetic case, the integro-local theorems are in fact the local ones: they concern approximating probabilities $\mathbf{P}(S_n = x)$ for $x \in \mathbb{Z}$).

The integro-local theorem is perhaps the most perfect and precise version of the classical central limit theorem. Indeed, it does not assume any additional conditions on top of the standard requirement of finite second moments (except for distinguishing between the lattice and non-lattice cases), but has basically got the same accuracy as the local limit theorems (note that, for small Δ, the left-hand side of (1) is "almost the density" of S_n), without making any assumptions about existence of the densities. That means that the integro-local theorems are much more precise than the integral ones, and it is easy to see that one can derive the assertions of the latter from the former, but not the other way around.

The integro-local theorems are rather effective and often the most adequate technical tools in a number of problems in probability theory. For instance, they are used for computing the exact asymptotics of large deviation probabilities for sums of independent r.v.'s (cf. Ch. 9 in Borovkov, 2013). They also proved instrumental for studying the distribution of the first passage time of a curvilinear boundary by a random walk (Borovkov, 2016a), establishing integro-local theorems for compound renewal processes (see Ch. 10 in Borovkov, 2013) and in a number of other problems.

Concerning the history of the problem, note that a special case of relation (1) (when x is fixed) was first established in Shepp (1964). A textbook exposition of the proof of (1) can be found in Section 8.7 of Borovkov (2013). Under additional Cramér's conditions (the moment generating function is finite in a neighborhood of zero and the strong non-lattice condition on the characteristic function of X is met, see (2) below), relation (1) was extended in Borovkov and Mogulskii (1999, 2001) in the multivariate setting to an asymptotic expansion in the powers of $n^{-1/2}$ and also to the case where Δ_0 can be vanishing. Extensions of Stone's theorem to the case of non-identically distributed independent r.v.'s in the triangular array scheme (covering the large deviations zone as well) were established in Borovkov (2011).

In the lattice case, an analog of (1) was obtained by B.V. Gnedenko in the univariate case (see Ch. 9 in Gnedenko and Kolmogorov, 1954) and by Rvačeva (1962) in the multivariate case.

It is most natural to ask if the remainder term in (1) can be sharpened under minimal additional assumptions. The first step in that direction was made in Borovkov (2016b), where it was shown that, under Cramér's strong non-lattice condition

$$\limsup_{|\lambda|\to\infty} |\varphi(\lambda)| < 1 \tag{2}$$

on the characteristic function (ch.f.) $\varphi(\lambda) := \mathbf{E}\, e^{i\lambda X},\ \lambda \in \mathbb{R}$, of X, and the moment condition $\mathbf{E}\,|X|^r < \infty$ for some $r \in (2, 3]$, relation (1) holds with the reminder term replaced by $O(\Delta n^{-(r-1)/2})$ uniformly in $x \in \mathbb{R}$ and in $\Delta \in (q^n, cn^{(3-r)/2})$ for some $q \in (0, 1)$ and every fixed $c > 0$. In fact, Borovkov (2016b) actually establishes a multivariate version of that result.

In the present note, we further develop the approach from Borovkov (2016b) to derive the first term of the asymptotic expansion for $\mathbf{P}(S_n \in \Delta[x))$ with uniform bounds for the remainder term in the case when condition (2) is met and $\mathbf{E}\,|X|^r < \infty$ for some $r \in [3, 4]$. It will be seen from the proofs that, under appropriate moment conditions, one can extend these results to asymptotic expansions with more terms. However, since the very form of such expansions and their derivations are getting quite cumbersome, while technically they are not much different from the one-term case, we will restrict ourselves to presenting the latter only.

To formally state our main results, we will need some further notations. For $r \in (2, \infty)$ and $b \in (1, \infty]$, introduce the class $\mathcal{F}_{r,b}$ of distributions F on $\mathbb{R}$ satisfying the following moment conditions: for $X \sim F$, one has $\mathbf{E}\,X = 0$, $\mathbf{E}\,X^2 = 1$ and

$$\mathbf{E}\,|X|^r < b.$$

In particular, $\mathcal{F}_{r,\infty}$ is the class of all zero mean unit variance distributions with a finite rth absolute moment. For $F \in \mathcal{F}_{3,\infty}$, we set

$$\mu_3 := \mathbf{E}\,X^3.$$

Further, for $\rho \in (0, 1]$ and $b < \infty$, we denote by $\mathcal{F}^{\rho}_{r,b}$ the totality of distributions from $\mathcal{F}_{r,b}$ that satisfy

$$\sup_{|\lambda|>1/b} |\varphi(\lambda)| < \rho. \tag{3}$$

When $b = \infty$, we will understand by $\mathcal{F}^1_{r,\infty}$ just the totality of distributions from $\mathcal{F}_{r,\infty}$ that satisfy Cramér's strong non-lattice condition (2) (or, equivalently, $\sup_{|\lambda|>\varepsilon} |\varphi(\lambda)| < 1$ for any $\varepsilon > 0$).

Theorem 1. (i) *For any distribution $F \in \mathcal{F}^1_{3,\infty}$, one has*

$$\frac{1}{\Delta}\mathbf{P}\big(S_n \in \Delta[x)\big) = \frac{1}{n^{1/2}}\,\phi\Big(\frac{x}{n^{1/2}}\Big)\Big(1 + \frac{\mu_3 x}{6n}\Big(\frac{x^2}{n} - 3\Big) - \frac{\Delta x}{2n}\Big) + \frac{R_n}{n}, \tag{4}$$

where for the remainder term $R_n = R_n(x, \Delta)$ the following holds true: there exists a $q \in (0, 1)$ such that, for any fixed $\Delta_1 > 0$,

$$\lim_{n\to\infty} \sup_{q^n \le \Delta \le \Delta_1} \sup_{x\in\mathbb{R}} R_n(x, \Delta) = 0.$$