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Strong stationary duality for discrete time Möbius monotone Markov chains on $\mathbb{Z}_+^d$

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ABSTRACT

For the discrete time Markov chains on $\mathbb{Z}_+^d$, we construct their strong stationary duality under the assumption of Möbius monotonicity. We also present an application to the two-dimensional birth and death chain.

1. Introduction and main results

Let $\mathbf{X} = (X_n)_{n\geq 0}$ be a time homogeneous Markov chain on the countable state space E, with initial distribution π_0 and transition matrix P. We write $\mathbf{X} \sim (\pi_0, P)$ for short below. Suppose $\mathbf{X}$ is ergodic with stationary distribution π. A strong stationary time (SST) introduced by Aldous and Diaconis (1986, 1987) is a randomized stopping time T for the chain $\mathbf{X}$ such that X_T has the stationary distribution π and is independent of T. The SST is a powerful tool to deal with the convergence to stationarity for a Markov chain in the sense of separation $s(n)$, which is given by $s(n) = \sup_{x\in E}[1 - \pi_n(x)/\pi(x)]$, where π_n is the distribution of X_n for $n \geq 0$. To get the SST, an effective way is to construct strong stationary dual (SSD) chain, which was introduced by Diaconis and Fill (1990a). They suggested a way to construct SSD chain on a discrete countable state space. Since the absorption time of the SSD chain is equal in distribution to an SST for $\mathbf{X}$, one can bound the SST in the original chain through the absorption time.

Later, on the linearly ordered state space, they showed a tractable case in Diaconis and Fill (1990a, Theorem 4.6). In this case, under the assumption of stochastic monotonicity for the time reversed chain, they showed how to construct SSD chain on the same state space. See Diaconis and Fill (1990b) and Fill (1991) for the extended theory of SSD.

Recently, Lorek and Szekli (2012, 2016) generalized the above mentioned construction from the linearly ordered state space to the finite partially ordered state space. In the place of stochastic monotonicity, the time reversed chain was assumed to be Möbius monotone.

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In this paper, we construct SSD for discrete time Möbius monotone Markov chains on $\mathbb{Z}_+^d$. Let us first recall the terminology in Diaconis and Fill (1990a,b), and give background material on duality.

Let $\mathbf{X} \sim (\pi_0, P)$ on E and $\mathbf{X}^* \sim (\pi_0^*, P^*)$ on the countable state space E^*. Assume $\mathbf{X}^*$ has an absorbing state a^*, let $\Lambda(x, y), x \in E^*, y \in E$, be a link or a transition kernel such that $\Lambda(a^*, \cdot) = \pi$. From Diaconis and Fill (1990b), we know that $\mathbf{X}^*$ is an SSD chain for $\mathbf{X}$ with respect to Λ if and only if the algebraic duality equations

$$\pi_0 = \pi_0^* \Lambda \quad \text{and} \quad \Lambda P = P^* \Lambda$$

hold. Moreover, the absorbing time $T_{a^*}^*$ that $\mathbf{X}^*$ hits a^* is an SST for $\mathbf{X}$.

Let $E = \mathbb{Z}_+^d$ with the natural partial order $\preceq$, i.e. for $x, y \in E, x \preceq y \iff x_i \le y_i, \forall\, 1 \le i \le d$. Following Lorek and Szekli (2012), we give the following definitions.

Definition 1.1. A nonnegative function f on E is Möbius monotone, if there exists a nonnegative function m on E such that $f(x) = \sum_{y \in E: y \succeq x} m(y)$ for $x \in E$.

Definition 1.2. A transition matrix P on $E \times E$ is Möbius monotone, if Pf is Möbius monotone whenever so is f.

To construct the dual chain, we will use the completion of the original state space E. That is $E^* = \overline{\mathbb{Z}}_+^d$ with $\overline{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{\infty\}$. The partial order $\preceq$ on E can be naturally extended to E^*, and we can also similarly define the Möbius monotonicity of nonnegative functions and transition matrices on E^*.

For a function f and a transition matrix P on E, let $P(x, \{y\}^{\downarrow}) = \sum_{z \in E: z \preceq y} P(x, z)$ for $x \in E, y \in E^*$. Fix $z \in E^*$, assume $\lim_{y \to x} f(y)$ and $\lim_{y \to x} P(y, \{z\}^{\downarrow})$ exist for $x \in E^* \setminus E$, then we define

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \in E; \\ \lim\limits_{y \to x} f(y), & \text{if } x \in E^* \setminus E, \end{cases}$$

and

$$\widetilde{P}(x, \{z\}^{\downarrow}) = \begin{cases} P(x, \{z\}^{\downarrow}), & \text{if } x \in E; \\ \lim\limits_{y \to x} P(y, \{z\}^{\downarrow}), & \text{if } x \in E^* \setminus E. \end{cases}$$

Now we can state our main theorem.

Theorem 1.3. *Let $\mathbf{X} \sim (\pi_0, P)$ be an ergodic Markov chain on $E = \mathbb{Z}_+^d$, with stationary distribution π. Define the time reversal of P as $\overleftarrow{P}(x, y) = \frac{\pi(y)}{\pi(x)} P(y, x)$, let $H(x) = \sum_{y \in E: y \preceq x} \pi(y)$ for $x \in E^*$ and $g(x) = \frac{\pi_0(x)}{\pi(x)}$. Then there exists an SSD chain $\mathbf{X}^* \sim (\pi_0^*, P^*)$ on $E^* = \overline{\mathbb{Z}}_+^d$ with respect to the link $\Lambda(x, y) = I_{(y \preceq x)} \frac{\pi(y)}{H(x)}$ if and only if the following two conditions hold:*

(a) *$\lim_{y \to x} g(y)$ exists for $x \in E^* \setminus E$ and $\widetilde{g}$ is Möbius monotone on E^*;*

(b) *for $x \in E^*$ and $z \in E^* \setminus E$, $\lim_{y \to z} \overleftarrow{P}(y, \{x\}^{\downarrow})$ exists and $\widetilde{\overleftarrow{P}}(\cdot, \{x\}^{\downarrow})$ is Möbius monotone on E^*.*

In this case, $(\infty, \ldots, \infty)$ is an absorbing state of the SSD chain $\mathbf{X}^* \sim (\pi_0^*, P^*)$, which is uniquely determined by

$$\frac{\pi_0(y)}{\pi(y)} = \sum_{z \in E^*: z \succeq y} \frac{\pi_0^*(z)}{H(z)} \quad \text{and} \quad \overleftarrow{P}(y, \{x\}^{\downarrow}) = \sum_{z \in E^*: z \succeq y} \frac{H(x)}{H(z)} P^*(x, z)$$

for $x \in E^*, y \in E$. Moreover, π_0^* and P^* are given explicitly by (3.2) and (3.4).

From Proposition 3.2 of Aldous and Diaconis (1987), we get the following proposition.

Proposition 1.4. *If T is an SST, then $s(n) \le \mathbb{P}(T > n)$ for $n \ge 0$.*

Useful bounds on total variation can be obtained by stopping the process before absorption. The early stopping theorem in Diaconis and Fill (1990a, Theorem 2.55) yields useful bounds on total variation. An argument similar to Diaconis and Fill (1990b, Corollary 2.1) shows the following corollary, which specializes the early stopping theorem to the chain $(\mathbf{X}^*, \mathbf{X})$ of Theorem 1.3, and transforms analysis of convergence rate for $\mathbf{X}$ into the first passage time for $\mathbf{X}^*$. Let π_n be the law of X_n. The total variation is defined by $\|\pi_n - \pi\| = \max_{A \subset E} |\pi_n(A) - \pi(A)|$.

Corollary 1.5. *For the chain $(\mathbf{X}^*, \mathbf{X})$ of Theorem 1.3, let T_A^* be the first hitting time of $A \subset E^*$ for $\mathbf{X}^*$, where $A = \{x \in E^* : x \succeq x^*\}$, $x^* \in E^*$. Then*

$$\|\pi_n - \pi\| \le (1 - H(x^*)) + H(x^*) \mathbb{P}\{T_A^* > n\}.$$

Here is the outline for the paper. Section 2 studies the *Möbius function* μ of E, and gives criteria on the Möbius monotonicity of nonnegative functions on E^*. Section 3 is devoted to the proof of Theorem 1.3. Section 4 shows an example of the two-dimensional birth and death chain.