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Strong laws for sequences in the vicinity of the LIL

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ABSTRACT

The present paper is devoted to strong laws of large numbers under moment conditions near those of the law of the iterated logarithm (LIL) for i.i.d sequences. More precisely, we wish to investigate possible limit theorems under moment conditions which are stronger than p for any $p < 2$, in which case we know that there is a.s. convergence to 0, and weaker than $EX^2 < \infty$, in which case the LIL holds.

1. Introduction and main result

Let $\{S_n,\ n \geq 1\}$ be the partial sums of independent, identically distributed (i.i.d.) random variables $\{X_k,\ k \geq 1\}$. The Kolmogorov and Marcinkiewicz and Zygmund (1937) strong laws tell us that, for $0 < p < 2$, $S_n/n^{1/p} \overset{a.s.}{\to} 0$ iff $E|X|^p < \infty$ (and the mean is zero whenever finite); see e.g. Gut (2013, Chapter 6). Hartman and Wintner (1941) proved the law of the iterated logarithm under the assumption of finite variance. Strassen (1966) proved that finite variance is a necessary condition (cf. e.g. Gut, 2013, Chapter 8).

The present paper is devoted to limit theorems "in the vicinity" of the law of the iterated logarithm for sequences, that is, to limit theorems under moment conditions which are stronger than p for any $p < 2$, in which case we know that there is a.s. convergence to 0, and weaker than finite variance, in which case the LIL holds. A companion paper devoted to the analogous problem for triangular arrays is our paper (Gut and Stadtmüller, submitted for publication). In a sequel to the present one, (Gut and Stadtmüller, in press), we prove analogs related to Cesàro summation.

Before we state our main result, we need some notation: $\log^+ x = \max\{1, \log x\}$, $g \in \mathcal{SV}$ means that the function g is slowly varying; for exact definitions and elementary properties one may consult (Bingham et al., 1987) or the Appendix in Gut (2013). Finally, C denotes a constant that may vary between appearances.

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Here is now our main result:

Theorem 1.1. *Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with mean 0 and partial sums S_n, $n \geq 1$. Further, let $\phi(x)$ be a positive, nondecreasing and unbounded function, such that $x/\phi(x)$ is eventually nondecreasing, and that, for some $\gamma, \ \rho > 0$,*

$$\frac{\phi(x)}{\phi(x\phi(x))} \geq \gamma > 0 \quad \text{and} \quad \phi(x) = \mathcal{O}\big((\log x)^{\rho}\big) \quad \text{as } x \to \infty. \tag{1.1}$$

If

$$\kappa := E\Big(\frac{X^2}{\phi(X^2)}\Big) < \infty, \tag{1.2}$$

then

$$\frac{S_n}{\sqrt{n \log\log n\, \phi(n)}} \overset{a.s.}{\to} 0 \quad \text{as } n \to \infty. \tag{1.3}$$

Conversely, if (1.3) holds, then

$$\sum_{n=3}^{\infty} P\big(|X| > \varepsilon\sqrt{n \log\log n\, \phi(n)}\,\big) < \infty \quad \text{for all } \varepsilon > 0. \tag{1.4}$$

If, moreover, $\phi(\cdot)$ satisfies (1.1), then

$$E\Big(\frac{X^2}{\log^+ \log^+ |X|\, \phi(X^2)}\Big) < \infty. \tag{1.5}$$

Remark 1.1. (i) Unfortunately we did not succeed in obtaining an equivalence between the strong law and the moment condition.

(ii) For typical ϕ-functions, such as logarithms or iterated logarithms, for example $\phi(x) = (\log^+ x)^{\alpha}/(\log^+ \log^+ x)^{\beta}$, where $\alpha > 0$ and $\beta \in \mathbb{R}$, Condition (1.2) is equivalent to the assumption that $E\,X^2/\phi(|X|) < \infty$.

(iii) The first part of Condition (1.1) is needed in order for the function $x\,\phi(x)$ to be invertible in a certain way. Strictly speaking we need (1.2) and $\sum_{n=3}^{\infty} P(|X| > \varepsilon\sqrt{n\,\phi(n)}) < \infty$, respectively, for the sufficiency. Lemma 2.1 shows that the two are equivalent under Condition (1.1).

(iv) The first part of Condition (1.1) is satisfied for functions increasing like log-powers. A limiting example is $\phi(x) = \exp\{(\log x)^{\alpha}\}$ for $\alpha \leq 1/2$. Functions such as $\phi(x) = (\log^+ x)^{\alpha}(\log^+ \log^+ x)^{\beta}$ with $\alpha, \ \beta \geq 0$ satisfy both assumptions in (1.1). □

Example 1.1. (i) Suppose that $E\,X^2/\log^+ |X| < \infty$, and let $\phi(x) = \log^+ x$. Then

$$\frac{S_n}{\sqrt{n \log n \log\log n}} \overset{a.s.}{\to} 0 \quad \text{as } n \to \infty.$$

This case could be reconsidered with Egorov (1972); cf. also Petrov (1995, Theorem 6.13 and notes, p. 226), and Petrov (1975, Problem 13, p. 289), where $g(x) = x^2/\phi(x^2) = x^2/(2\log^+ x)$. For a function $f(n)$ satisfying $\sum_n \frac{1}{n(f(n))^k} < \infty$ for some $k \geq 1$, this leaves us with, e.g., $f(n) = (\log n)^{1/j}$ with $j < k$, and the slightly weaker statement

$$\frac{S_n}{\sqrt{n \log n (\log(n \log n))^{1/j}}} \sim \frac{S_n}{\sqrt{n (\log n)^{1+1/j}}} \overset{a.s.}{\to} 0 \quad \text{as } n \to \infty.$$

(ii) Suppose that $E\,X^2/(\log^+ \log^+ |X|)^{\alpha-1} < \infty$, and let $\phi(x) = (\log^+ \log^+ x)^{\alpha-1}$, $\alpha > 1$. Then

$$\frac{S_n}{\sqrt{n (\log\log n)^{\alpha}}} \overset{a.s.}{\to} 0 \quad \text{as } n \to \infty.$$

(iii) Suppose that $E\,X^2/(\log^+ \log^+ \log^+ |X|)^{\beta} < \infty$, and let $\phi(x) = (\log^+ \log^+ x)^{\beta}$, $\beta > 0$. Then

$$\frac{S_n}{\sqrt{n \log\log n (\log\log\log n)^{\beta}}} \overset{a.s.}{\to} 0 \quad \text{as } n \to \infty. \quad \square$$

Remark 1.2. A relevant reference here is Einmahl and Li (2005), who, using a different approach, study this and related problems under slightly different assumptions. □