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Identifiability of nonrecursive structural equation models

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ABSTRACT

An approach to studying reciprocal-effects in a cross-sectional data is to apply nonrecursive structural equation models. This article takes a theoretical approach to study the identifiability problem of reciprocal-effect models. Identifiability conditions are presented under a variety of settings.

1. Introduction

Causal analysis is a primary theme of science, and empirical and theoretical arguments are both important. There are several different types of causal relations, and among these, reciprocal relationships are the most difficult to identify, where events relate reciprocally and directly to each other, i.e., an event concurrently takes on the roles of cause and outcome. Once a model with reciprocal relationship has been identified and estimated, one can distinguish among the three models, namely, $A \rightleftarrows B, A \rightarrow B$ and $A \leftarrow B$ for events A and B (see e.g., Richardson, 1996; Shimizu et al., 2006; Lacerda et al., 2008). A structural equation model (SEM) is said to be nonrecursive if it includes reciprocal relationships or feedback loops, and/or some disturbances are correlated (Paxton et al., 2011). Nonrecursive models are often not identified and cannot be estimated. Introducing instrumental variables that satisfy certain specific conditions plays an important role in achieving identification. As will be discussed below, it is also important to study effects of covariation between disturbance terms on identification of the model, in addition to instrumental variables. In the literature, there are very few reports of studies that have considered the following: (1) *identifiability* conditions for models with more than two variables with reciprocal relations; (2) how disturbance terms affect the conditions for *identifiability* in nonrecursive SEMs; and (3) how nonrecursive SEMs are related to similar models, such as the spatial or network autocorrelation models that have been developed in the fields of spatial econometrics (Lesage and Pace, 2004) and social influences (Leenders, 2002).

The aim of this article is to study those problems mathematically in order to study the nature of the nonrecursive models.

2. Model setting

Let $\boldsymbol{y}$ and $\boldsymbol{x}$ be $p \times 1$ and $q \times 1$ random vectors of observations, respectively. There may be reciprocal relationships between several variables within the vector $\boldsymbol{y}$. The variables in vector $\boldsymbol{x}$ are usually referred to as instrumental variables, and they

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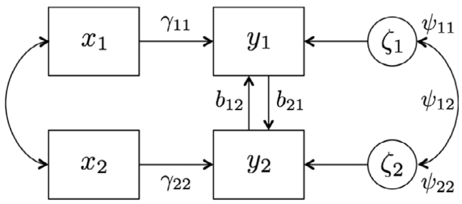

Fig. 1. Model 1 ($p = 2,\ q = 2$).

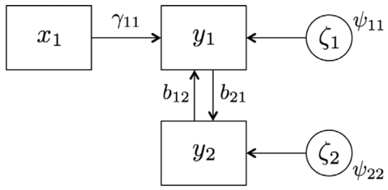

Fig. 2. Illustrative model 2 ($p = 2,\ q = 1$).

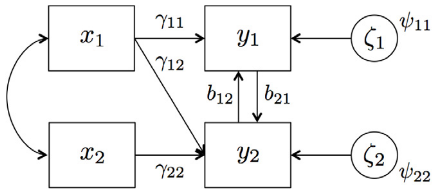

Fig. 3. Illustrative model 3 ($p = 2,\ q = 2$).

must satisfy certain conditions. The structural equation for this model is

$$\boldsymbol{y} = \boldsymbol{B}\boldsymbol{y} + \boldsymbol{\Gamma}\boldsymbol{x} + \boldsymbol{\zeta} \ \Leftrightarrow\ \boldsymbol{y} = (\boldsymbol{I} - \boldsymbol{B})^{-1}(\boldsymbol{\Gamma}\boldsymbol{x} + \boldsymbol{\zeta}). \tag{1}$$

Here, the reciprocal relationships are described by the $p \times p$ matrix $\boldsymbol{B} = (b_{ij})$, and the diagonal elements of $\boldsymbol{B}$ are fixed at 0. The $p \times q$ matrix $\boldsymbol{\Gamma}$ consists of the regression coefficients from $\boldsymbol{x}$ to $\boldsymbol{y}$, $\boldsymbol{I} - \boldsymbol{B}$ is assumed to be nonsingular, and the maximum of absolute eigenvalue of $\boldsymbol{B}$ is less than 1. The $p \times 1$ vector $\boldsymbol{\zeta}$ comprises random terms which are usually assumed to be independent of $\boldsymbol{x}$, and we denote $\mathrm{Cov}[\boldsymbol{\zeta}] = \boldsymbol{\Psi}$, $\mathrm{Cov}[\boldsymbol{x}] = \boldsymbol{\Phi}$. The implied variance and covariance matrix derived from Eq. (1) can be expressed as

$$\boldsymbol{\Sigma}(\boldsymbol{\theta}) = \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{B})^{-1}\{\boldsymbol{\Gamma}\boldsymbol{\Phi}\boldsymbol{\Gamma}' + \boldsymbol{\Psi}\}(\boldsymbol{I} - \boldsymbol{B})^{-1'} & (\boldsymbol{I} - \boldsymbol{B})^{-1}\boldsymbol{\Gamma}\boldsymbol{\Phi} \\ \boldsymbol{\Phi}\boldsymbol{\Gamma}'(\boldsymbol{I} - \boldsymbol{B})^{-1'} & \boldsymbol{\Phi} \end{bmatrix}, \tag{2}$$

where the parameter $\boldsymbol{\theta}$ consists of all the elements of $\boldsymbol{B}$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Psi}$ and , $\boldsymbol{\Phi}$.

2.1. Basic models for illustration

The model in Fig. 1 is known to be identifiable (Bollen, 1989). Here, the instrumental variables $\boldsymbol{x} = (x_1, x_2)$ have to satisfy that there are no direct effects from x_1 to y_2 and from x_2 to y_1, in other words, $\gamma_{12} = \gamma_{21} = 0$.

The parameter matrices $\boldsymbol{B}$, $\boldsymbol{\Gamma}$, and $\boldsymbol{\Psi}$ for this model are

$$\boldsymbol{B} = \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \end{bmatrix}, \qquad \boldsymbol{\Gamma} = \begin{bmatrix} \gamma_{11} & 0 \\ 0 & \gamma_{22} \end{bmatrix}, \qquad \boldsymbol{\Psi} = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{12} & \psi_{22} \end{bmatrix}.$$

If we assume there is no covariance between the errors, i.e., $\psi_{12} = \psi_{21} = 0$, then the models in Figs. 2 and 3 are also identified (Berry, 1984).

2.2. Autocorrelation model

Nonrecursive models fall roughly into two categories: those concerning causal effects, as discussed in Section 2.1, and those concerning neighbor interactions, which will be discussed here. A representative example of the latter would be a spatial autocorrelation model (Lesage and Pace, 2004) or network autocorrelation model (Leenders, 2002), such as is often used in the field of spatial econometrics or in social network analysis. In this type of model, the interdependence of the