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A scalar-valued infinitely divisible random field with Pólya autocorrelation

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A B S T R A C T

We construct and characterize a stationary scalar-valued random field with domain $\mathbb{R}^d$ or $\mathbb{Z}^d$, $d \in \mathbb{Z}^+$, which is infinitely divisible, can take any (univariate) infinitely divisible distribution with finite variance at any single point of its domain, and has autocorrelation function between any two points in its domain expressed as a product of arbitrary positive and convex functions equal to 1 at the origin. Our method of construction – based on carefully chosen sums of independent and identically distributed random variables – is simple and so lends itself to simulation.

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1. Introduction

Finlay et al. (2011) construct a stationary univariate stochastic process $\{V(t)\}$, $t \in \mathbb{R}$ or $t \in \mathbb{Z}$, that can possess any infinitely divisible distribution with finite variance at any time point t, and any Pólya- or Young-type autocorrelation function (Pólya- and Young-type autocorrelation functions are defined in Section 2 and essentially entail functions that are positive and convex; the allowable set of autocorrelation functions is given by Assumption 1). The basic idea of the construction is as follows: given a prescribed autocorrelation function $\rho(s)$, $V^n(t)$ is defined from the sum of n independent and identically distributed (i.i.d.) random variables such that the number of such variables common to $V^n(t)$ and $V^n(t+s)$ is $\lfloor n\rho(s + (nt - \lfloor nt \rfloor)/n)\rfloor$ for $\lfloor \cdot \rfloor$ denoting the integer part. Taking the limit as $n \to \infty$, $\mathbb{C}\mathrm{or}(V(t), V(t+s)) = \rho(s)$.

Finlay and Seneta (2014) extend this result to a stationary multivariate stochastic process; that is, to a process $\{V(t)\} = \{V_1(t), \ldots, V_d(t)\}$, $d \in \mathbb{Z}^+$, $t \in \mathbb{R}$ or $t \in \mathbb{Z}$, which again can have any specified infinitely divisible distribution with finite variance, the same for each random variable $V_h(t)$ for fixed h and t, and any Pólya- or Young-type autocorrelation function $\rho(s)$ for any pair of random variables $V_h(t)$ and $V_i(t+s)$.

In this article we extend Finlay et al. (2011) in a different direction: to a stationary infinitely divisible random field with range again in $\mathbb{R}$ (that is, scalar-valued like Finlay et al., 2011, but unlike Finlay and Seneta, 2014), but domain now defined on $\mathbb{R}^d$ or $\mathbb{Z}^d$, $d \in \mathbb{Z}^+$. Section 3 constructs a random field defined on $\mathbb{R}^d$ with Pólya-type autocorrelation function, while Section 4 constructs a random field defined on $\mathbb{Z}^d$ with Young-type autocorrelation function. To be precise, and focusing on the continuous time case, we construct a scalar-valued random field $V_d : \mathbb{R}^d \to \mathbb{R}$ for $d \in \mathbb{Z}^+$, denoted by $\{V_d(t_d)\}$, $t_d \in \mathbb{R}^d$. Importantly, the value of V_d evaluated at any t_d is a random scalar with distribution that can be specified as any infinitely divisible distribution. For $t_d, s_d \in \mathbb{R}^d$ the correlation structure (assuming finite variance) is given

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by $\mathbb{C}\mathrm{or}(V_d(t_d), V_d(s_d)) = \prod_{k=1}^{d} \rho_k(t_d[k] - s_d[k])$ where $t_d[k]$ denotes the kth element of t_d, and each ρ_k may be specified as any non-negative convex function that is equal to 1 at the origin (this is the Pólya-type condition).

We also provide the characteristic function of the finite dimensional distributions over the domain of definition of this scalar random field, and in the process show that these are infinitely divisible, so that the random field itself is infinitely divisible. This thus extends the results described above even in the case $d = 1$.

For $d = 1$ our random field reduces to the stochastic process defined on $\mathbb{R}$ from Finlay et al. (2011), where the marginal distribution of the stochastic process evaluated at any point may be taken as any infinitely divisible distribution, and the correlation structure for $t_1, s_1 \in \mathbb{R}$ is given by $\mathbb{C}\mathrm{or}(V_1(t_1), V_1(s_1)) = \rho_1(t_1 - s_1)$, while for $d = 2$ we have a random sheet with domain on $\mathbb{R}^2$ (see for example Kroese and Botev, 2015), where again the marginal distribution of the stochastic process evaluated at any point may be taken as any infinitely divisible distribution, and the correlation structure for $t_2, s_2 \in \mathbb{R}^2$ is given by $\mathbb{C}\mathrm{or}(V_2(t_2), V_2(s_2)) = \rho_1(t_2[1] - s_2[1]) \times \rho_2(t_2[2] - s_2[2])$.

Our contribution to the literature is to construct and characterize a class of flexible (in terms of distribution and correlation structure) non-Gaussian random field for possible use in modeling and estimation; our method of construction, based on sums of i.i.d. random variables, also lends itself particularly easily to simulation, while the random field is fully characterized in terms of the characteristic function of its finite dimensional distributions, allowing for efficient estimation. This complements the work of many others who have constructed non-Gaussian random fields: see for example Barndorff-Nielsen et al. (2014) who construct ambit fields for use in modeling electricity futures, Biermé et al. (2007) who characterize a class of operator-scaling stable random fields, and Karcher et al. (2013) who provide a simulation method for infinitely divisible random fields, as well as a number of papers by Ma (2009, 2011a,b,c,d). Cressie (2015) provides an introduction to the closely related area of spatial statistics.

Of course the class of admissible covariance functions for stationary scalar-valued Gaussian random fields, as well as other closely related random fields such as elliptically contoured random fields, is well-known (an elliptically contoured random field is essentially a Gaussian random field multiplied by a non-negative random variable; the marginal distribution of the resulting random field is altered by the multiplication but the correlation structure is not). In this case the covariance function c may be taken as any function that satisfies

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i c(t_i, t_j) a_j \geq 0 \tag{1}$$

for all a_k and t_k (see for example Cramér and Leadbetter, 1967, Feller, 1966 and Gikhman and Skorokhod, 1969, as well as Ma, 2009, 2011a,b,c,d and the references therein). For random fields that are non-Gaussian, however, Eq. (1) is in general a necessary but not a sufficient condition for a covariance function, and the range of admissible covariance structures must be investigated on a case-by-case basis.

2. Pólya- and Young-type autocorrelation functions

Pólya (1949) provides a simple sufficient condition for the admissibility of a continuous time autocorrelation function of a univariate process, being essentially that a function $\rho(s)$ is admissible if it is real-valued, continuous and symmetric about the origin, with $\rho(0) = 1$, $\rho(s)$ convex for $s > 0$ and $\rho(s) \to 0$ as $s \to \infty$. This Pólya condition is useful in the univariate setting as it is reasonably flexible and importantly is easy to check in practice. There is a more general necessary and sufficient condition as given by Eq. (1), but its practical use is more limited as for a given $\rho(s)$ it can be difficult to check.

In fact Pólya's condition was originally stated in terms of characteristic functions, but a function is a real-valued characteristic function if and only if it is also an admissible autocorrelation function (see for example Finlay et al., 2011 for this equivalence, and Christakos, 1984 for a similar condition to Pólya's).

A related theorem from Young (1913) gives an analogous result for the discrete time setting. For $\rho(s)$, $s \in \mathbb{Z}$, Young's theorem essentially states that $\rho(s)$ is an admissible discrete time autocorrelation function if it is real-valued and symmetric on $\mathbb{Z}$ with $\rho(0) = 1$, $\rho(s) \to 0$ as $s \to \infty$, and $\rho(s) \geq 0$, $\rho(s+1) - \rho(s) \leq 0$, $\rho(s+2) - 2\rho(s+1) + \rho(s) \geq 0$ for $s = 0, 1, 2, \ldots$ (see also Chapter V of Zygmund, 1968, as well as Kolmogoroff, 1923). Similar to the Pólya condition, the result was originally stated in the context of Fourier series, but the Fourier series can be interpreted as a symmetric probability density function on $(-\pi, \pi)$ and, inverting the Fourier series, the $\rho(s)$ for $s \in \mathbb{Z}$ (the Fourier coefficients) can be interpreted as the characteristic function of this probability density function evaluated at the integers. Being the characteristic function of a symmetric density function, and so real-valued, $\rho(s)$ is also an admissible discrete time autocorrelation function.

These Pólya and Young sufficient conditions turn out to very nearly define the set of autocorrelation functions possible using the method that we employ; our method essentially involves constructing random fields via carefully chosen sums of i.i.d. random variables, and the Pólya (continuous setting) and Young (discrete setting) conditions ensure that all sums that we consider are non-negative. (Note, however, that we do not require that $\rho(s) \to 0$ as $s \to \infty$, but can relax this to $\rho(s) \to \delta \in [0, 1]$ as $s \to \infty$.)

3. A random field in continuous time

We proceed with the construction by induction: we first assume that an approximation to the random field $\{V_{d-1}(t_{d-1})\}$ for $t_{d-1} \in \mathbb{R}^{d-1}$ exists with the specified distribution and correlation structure, and show that this implies that an

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