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The limit law of the iterated logarithm for linear processes

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ABSTRACT

In this short note, we established the limit law of the iterated logarithm for linear process. Let $\{\xi_i, -\infty < i < \infty\}$ be a sequence of independent identically distributed random variables with $E\xi_1 = 0$ and $E\xi_1^2 = 1$. Define the linear process by $X_t = \sum_{j=-\infty}^{\infty} a_j \xi_{t-j}, t \geq 1$ and the partial sum $S_n = \sum_{t=1}^{n} X_t$, where $\{a_j, -\infty < j < \infty\}$ is a sequence of real numbers with $\sum_{j=-\infty}^{\infty} a_j \neq 0$ and $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Then, we have

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\log\log n}} \max_{1\leq k\leq n} \frac{|S_k|}{\sqrt{k}} = \left|\sum_{j=-\infty}^{\infty} a_i\right| \quad a.s.$$

1. Introduction and main results

The linear processes are important in time series analysis and have many applications to other fields, like economics, engineering, and physical science. A vast amount of literature is devoted to the study of the theorems for linear processes under various conditions. In this short note, we intend to establish the limit law of the iterated logarithm (LIL) for linear process under finite second moment condition.

The LIL is an important aspect in probability theory because it describes the precise convergence rates. Suppose that $\{X, X_k, k \geq 1\}$ is a sequence of i.i.d. random variables and $S_n = \sum_{k=1}^{n} X_k$, it is well known that Hartman–Wintner–Strassen law of iterated logarithm (LIL) states that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = \sigma \quad \text{a.s.} \tag{1.1}$$

if and only if

$$EX = 0 \quad \text{and} \quad EX^2 = \sigma^2 < \infty. \tag{1.2}$$

We refer Hartman and Wintner (1941) for the "if" part and Strassen (1966) for the "only if" part. The above LIL is also regarded as the limsup LIL. The liminf LIL was established by Chung (1948) who prove that

$$\liminf_{n\to\infty} \sqrt{\frac{\log\log n}{n}} \max_{1\leq k\leq n} |S_k| = \frac{\pi}{\sqrt{8}}\sigma \quad \text{a.s.} \tag{1.3}$$

under the extra assumption of $E|X|^3 < \infty$. Jain and Pruitt (1975) show that (1.2) is sufficient and necessary for (1.3).

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Recently, Chen (2015) established the following limit LIL for the partial sums

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\log\log n}} \max_{1\le k\le n} \frac{S_k}{\sqrt{k}} = \sigma \quad \text{a.s.} \tag{1.4}$$

under the assumption (1.2). Later on, Li and Liang (2013) obtained a version of the limit LIL in Banach space, Fu et al. (2015) established the limit LIL for B-valued trimmed sums.

For linear processes $X_t = \sum_{j=0}^{\infty} a_j \xi_{t-j}$, $t \ge 1$, where $\{\xi_i, -\infty < i < \infty\}$ is a sequence of random variables and $\{a_i, 0 \le i < \infty\}$ is a sequence of real numbers. Phillips and Solo (1992) established the limsup LIL for the i.i.d. innovations with zero means and $E|\xi_1|^p < \infty$ for some $p > 2$ and $\sum_{j=1}^{\infty} j^2 a_j^2 < \infty$ or $\sum_{j=1}^{\infty} j^{1/2}|a_j| < \infty$. We also refer to Yokoyama (1995) for the limsup LIL with α-mixing innovations, Yang (1996) for the limsup LIL with i.i.d. innovations in Banach space, Lin and Li (2008) for the limsup LIL and liminf LIL with ϕ-mixing innovations, Tan et al. (2008) for the limsup LIL with negatively associated innovations. As we know, there are no results about the above limit LIL for linear processes. The purpose of this short note is to establish the following version of the limit LIL for linear processes with i.i.d. innovations.

Theorem 1.1. *Let $\{\xi_i, -\infty < i < \infty\}$ be a sequence of independent identically distributed random variables with $E\xi_1 = 0$ and $E\xi_1^2 = 1$. Define the linear process by $X_t = \sum_{j=-\infty}^{\infty} a_j \xi_{t-j}$, $t \ge 1$ and the partial sum $S_n = \sum_{t=1}^{n} X_t$, where $\{a_i, -\infty < i < \infty\}$ is a sequence of real numbers satisfying $A = \sum_{j=-\infty}^{\infty} a_j \neq 0$, $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Then*

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\log\log n}} \max_{1\le k\le n} \frac{|S_k|}{\sqrt{k}} = |A| \quad a.s., \tag{1.5}$$

where $\log n = \log(e \vee n)$ and $\log\log n = \log(\log n)$, $n \ge 1$.

Remark 1.2. If we take $a_0 = 1$ and $a_j = 0$ for $j \neq 0$, then we can obtain the result of Chen (2015). In Chen's paper, the proof depends on the almost sure invariance principle (strong approximation) for the partial sums of i.i.d. random variables. In fact, we can give the limit LIL for linear process with i.i.d. innovation by using the almost sure invariance principle for linear process, but it need more assumptions of $\{\xi_i\}$ and $\{a_i\}$ to get the suitable rate for the almost sure invariance principle of linear process (one can see Lin and Li, 2008, Lu and Qiu, 2007, Tan et al., 2008 for details). Therefore, we apply the Beveridge and Nelson decomposition for linear process to prove Theorem 1.1. Using this decomposition, we only need the finite second moments of $\{\xi_i\}$ and the absolutely summable assumption of $\{a_i\}$.

Remark 1.3. Theorem 1.1 can be extended to some other dependent or mixing innovations, such as negatively associated (NA), linearly negative quadrant dependent (LNQD), ρ-mixing, α-mixing, and so on. In fact, the key steps in the proof of Theorem 1.1 are Lemma 2.2 and (1.4). We claim that Lemma 2.2 and (1.4) still hold for NA, ρ-mixing, α-mixing innovations under some suitable conditions on the underlying sequences $\{\xi_i\}$ and $\{a_i\}$. By the exponential inequality (we can refer to Shao and Su, 1999 for NA, Wang et al., 2010 for LNQD, Shao, 1990 for ρ-mixing and α-mixing) and the same methodology in Lemma 2.2, one can see that Lemma 2.2 holds for the above sequences, (1.4) can be established by the same methodology of Chen's paper. However, the conditions may be complex and strict. So it is an interesting problem to find the optimal assumptions of $\{\xi_i\}$ and $\{a_i\}$ such that Lemma 2.2 and (1.4) still hold for dependent or mixing sequences and we will discuss the details in the future.

Throughout the sequel, C represents a positive constant although its value may change from one appearance to the next, $I\{A\}$ denotes the indicator function of the set A.

2. Proof of Theorem 1.1

The following two lemmas are useful for the proof of Theorem 1.1. The first one was introduced by Li and Liang (2013).

Lemma 2.1. *Let $\{a_n, n \ge 1\}$ be a nondecreasing sequence of positive real numbers such that $\lim_{n\to\infty} a_n = \infty$. Then for any real sequence $\{b_n, n \ge 1\}$, we have*

$$\limsup_{n\to\infty} \frac{1}{a_n} \max_{1\le k\le n} b_k = 0 \vee \limsup_{n\to\infty} \frac{b_n}{a_n}. \tag{2.1}$$

Lemma 2.2. *Let $\{\xi_k, -\infty < k < \infty\}$ be a sequence of independent identically distributed random variables with $E\xi_1 = 0$ and $E\xi_1^2 = 1$. Then*

$$E \sup_n (2n\log\log n)^{-\frac{1}{2}} \left| \sum_{k=1}^{n} \xi_k \right| < \infty. \tag{2.2}$$