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Weak convergence of h-transforms for one-dimensional diffusions

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ABSTRACT

It is proved that the h-transform of the killed process for one-dimensional diffusions with respect to the scale function is weakly continuous with respect to the starting point. A similar result is obtained for its bridges.

1. Introduction

Let $M = \{X, (P_x)_{x\geq 0}\}$ be the canonical representation of a one-dimensional diffusion process, where we assume that 0 is an instantaneously reflecting boundary as well as a regular recurrent state. We write S for the scale function of M, with $S(0) = 0$ and $S(\infty) = \infty$, m for the speed measure and $\mathcal{G} = \frac{d}{dm}\frac{d}{dS}$ for the canonical form of the infinitesimal generator of M. Following Biane and Yor (1987), we adopt the canonical path space as the set of real-valued continuous paths w which are defined on $[0, \zeta(w)]$ with lifetime $\zeta(w) \in (0, \infty)$ or on $[0, \infty)$ with $\zeta(w) = \infty$. Let $X = (X_t, t \geq 0)$ denote the coordinate process, $(\mathcal{F}_t, t \geq 0)$ the natural filtration and $H_x = \inf\{t : X_t = x\}$. Let $\mathbf{n}$ denote the Itô measure of M normalized by $\mathbf{n}(H_x < \zeta) = 1/S(x)$ for all $x > 0$.

We may define $M^\uparrow = \{X, (P_x^\uparrow)_{x\geq 0}\}$ as follows:

$$P_x^\uparrow(A; t < \zeta) = \frac{E_x[1_A S(X_t); t < H_0]}{S(x)} \quad (x > 0) \tag{1.1}$$

and

$$P_0^\uparrow(A; t < \zeta) = \mathbf{n}[1_A S(X_t); t < \zeta] \tag{1.2}$$

for $A \in \mathcal{F}_t$ and $t > 0$. (Note that $M^\uparrow$ is *not* conservative when $m(\infty) < \infty$; see Yano and Yano, 2015.) The process $M^\uparrow = \{X, (P_x^\uparrow)_{x\geq 0}\}$ appears in certain representations of the Itô measure and in conditioning to avoid zero; see Yano (2006), Salminen et al. (2007), Salminen et al. (2015) and Yano and Yano (2015). A natural question arises: does the following weak convergence hold?

$$P_x^\uparrow \xrightarrow{\text{w}} P_0^\uparrow \quad \text{as } x \downarrow 0. \tag{1.3}$$

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We also study the $M^\uparrow$-bridges. We denote by $P^{\uparrow,u}_{x,y}$ the law of the $M^\uparrow$-bridge of duration u, starting at x and ending at y, which may be defined via h-transforms (see Fitzsimmons et al., 1993, Chaumont and Uribe Bravo, 2011) as follows:

$$P^{\uparrow,u}_{x,y}(A) = E^\uparrow_x\left[1_A \frac{p^\uparrow_{u-t}(X_t, y)}{p^\uparrow_u(x, y)}; t < \zeta\right] \quad (x, y \geq 0), \tag{1.4}$$

for $A \in \mathcal{F}_t$ and $0 < t < u$, where $p^\uparrow_u(x, y)$ denotes the transition density of $M^\uparrow$ with respect to its speed measure. We may now raise a natural question: does the following weak convergence hold?

$$P^{\uparrow,u}_{x,y} \xrightarrow{\text{w}} P^{\uparrow,u}_{0,0} \quad \text{as } x, y \downarrow 0. \tag{1.5}$$

The statements (1.3) and (1.5) of weak convergence have been taken for granted in Salminen et al. (2007) and Salminen et al. (2015). The aim of this paper is to give proofs for (1.3) and (1.5). Note that similar problems have been discussed in different settings in Doney (2007) and Chaumont and Uribe Bravo (2011).

The organization of the paper is as follows. In Section 2, we recall basic facts about $M^\uparrow$ and its bridges. In Section 3, we prove weak convergence results for these processes.

2. Notation and basic facts

We need several notation and basic facts about $M^\uparrow$ and its bridges; The proofs of all the assertions in this section can be found in Yano (2006).

We write $\widehat{M} = \{X, (\widehat{P}_x)_{x\geq 0}\}$ for the process obtained by killing M at H_0, i.e.,

$$\widehat{P}_x(A; t < \zeta) = P_x(A; t < H_0) \tag{2.1}$$

for $A \in \mathcal{F}_t$ and $t > 0$. There exists a continuous density of the transition probability:

$$\widehat{P}_x(X_t \in dy; t < \zeta) = \widehat{p}_t(x, y)m(dy). \tag{2.2}$$

We may choose

$$m^\uparrow(dy) = (S(y))^2 m(dy), \qquad S^\uparrow(x) = -\frac{1}{S(x)}. \tag{2.3}$$

as the speed measure $m^\uparrow$ and the scale function $S^\uparrow$ for $M^\uparrow$, respectively. There exists a continuous density of the transition probability:

$$P^\uparrow_x(X_t \in dy; t < \zeta) = p^\uparrow_t(x, y)m^\uparrow(dy), \tag{2.4}$$

where the densities $p^\uparrow_t(x, y)$ and $\widehat{p}_t(x, y)$ are connected by the relation

$$p^\uparrow_t(x, y) = \frac{\widehat{p}_t(x, y)}{S(x)S(y)}, \quad x, y > 0. \tag{2.5}$$

Note that $p^\uparrow_t(x, y)$ may be continuously extended for $x, y \geq 0$ for all $t > 0$. We have the Chapman–Kolmogorov identity:

$$p^\uparrow_{t+s}(x, z) = \int_0^\infty p^\uparrow_t(x, y)p^\uparrow_s(y, z)m^\uparrow(dy), \quad t, s > 0, \ x, z \geq 0. \tag{2.6}$$

The boundary values of $p^\uparrow_t(x, y)$ for $x = 0$ or $y = 0$ play the following roles. The law of the lifetime of a generic excursion is given as

$$\mathbf{n}(\zeta \in dt) = p^\uparrow_t(0, 0)dt. \tag{2.7}$$

If we set

$$f_{x0}(t) := p^\uparrow_t(0, x)S(x) = p^\uparrow_t(x, 0)S(x) = \lim_{y\downarrow 0} \frac{\widehat{p}_t(x, y)}{S(y)}, \quad x > 0. \tag{2.8}$$

Then $f_{x0}(t)$ is a density of the hitting time of 0:

$$P_x(H_0 \in dt) = f_{x0}(t)dt \tag{2.9}$$

and at the same time it is a density of the entrance law of the Itô measure:

$$\mathbf{n}(X_t \in dx; t < \zeta) = f_{x0}(t)m(dx). \tag{2.10}$$