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A note on Marked Point Processes and multivariate subordination

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ABSTRACT

The aim of this paper is to state a correspondence between marked Poisson processes and multivariate subordinated Lévy processes. We prove that, under suitable conditions, marked Poisson processes are in law subordinated Brownian motions and we provide their Lévy triplet and characteristic function. We introduce the class of multivariate Gaussian marked Poisson processes and prove that – in law – they are multivariate subordinated Brownian motions.

0. Introduction

The aim of this note is to state a correspondence between multivariate marked Poisson processes and multivariate subordinated Lévy processes. The motivation of this note is to further the work of Barndorff-Nielsen et al. (2001) who give characterization of multivariate subordinated Lévy processes which are relevant in modern finance and related fields. A standard reference for Poisson processes, Lévy processes and their relationship is Çınlar (2011).

Let $\mathbf{\Pi}$ be a Poisson random measure on a measurable space $(E, \mathcal{E})$ with a σ-finite mean measure $\mu_{\mathbf{\Pi}}$. By slight abuse of notation, with $\mathbf{\Pi} = \{\Pi_i, i \in I\}$ we indicate both the random measure and the collection of its atoms indexed by some countable set I. Marked Poisson processes are constructed by attaching a random variable to each atom of the random measure $\mathbf{\Pi}$. Formally, let $\boldsymbol{Z} = \{Z_i,\ i \in I\}$ be a family of random variables (marks) on a measurable space $(F, \mathcal{F})$ indexed by the same countable set I. Assume that the variables Z_i are conditionally independent given $\mathbf{\Pi}$ with distributions $Q(\Pi_i, \cdot)$, where $Q(\boldsymbol{s}, B)$ is a transition probability kernel from $(E, \mathcal{E})$ into $(F, \mathcal{F})$. Each variable Z_i can be considered as an indicator of

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some property associated with the atom Π_i. Then, as proved in Theorem 3.2 in Çınlar (2011), $\boldsymbol{M} := (\boldsymbol{\Pi}, \boldsymbol{Z})$ forms a Poisson random measure on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ with mean $\mu_{\boldsymbol{\Pi}} \times Q$, where $(\mu_{\boldsymbol{\Pi}} \times Q)(d\boldsymbol{x}, d\boldsymbol{y}) = \mu_{\boldsymbol{\Pi}}(d\boldsymbol{x})Q(\boldsymbol{x}, d\boldsymbol{y})$. The new measure $\boldsymbol{M}$ is called marked Poisson random measure.

Let us recall that the subordination of a Lévy process $\boldsymbol{L} = \{\boldsymbol{L}(t), t \geq 0\}$ by a univariate subordinator $\tau(t)$, i.e. a Lévy process on $\mathbb{R}_+ = [0, \infty)$ with increasing trajectories, defines a new process $\boldsymbol{X} = \{\boldsymbol{X}(t), t \geq 0\}$ by the composition $\boldsymbol{X}(t) := (L_1(\tau(t)), \dots, L_n(\tau(t)))^T$. Theorem 30.1 in Sato (1999) characterizes the subordinated process $\boldsymbol{X}$ in terms of its Lévy triplet. Barndorff-Nielsen et al. (2001) generalize the above construction by allowing the introduction of multivariate subordinators, i.e. Lévy processes on $\mathbb{R}^n_+ = [0, \infty)^n$, whose trajectories are increasing in each coordinate. For purposes of introduction of multivariate subordination, the notion of $\mathbb{R}^d_+$-parameter process, as introduced in Barndorff-Nielsen et al. (2001), is required. Consider the multiparameter $\boldsymbol{s} = (s_1, \dots, s_d)^T \in \mathbb{R}^d_+$ and the partial order on $\mathbb{R}^d_+$

$$\boldsymbol{s}^1 \preceq \boldsymbol{s}^2 \;\Leftrightarrow\; s^1_j \leq s^2_j, \quad j = 1, \dots, d.$$

Let now $\boldsymbol{L}(\boldsymbol{s}) = (L_1(\boldsymbol{s}), L_2(\boldsymbol{s}), \dots, L_n(\boldsymbol{s}))^T$ be a process with parameters in $\mathbb{R}^d_+$ and values in $\mathbb{R}^n$. It is called an $\mathbb{R}^d_+$-parameter Lévy process on $\mathbb{R}^n$ if the following holds

- for any $m \geq 3$ and for any choice of $\boldsymbol{s}^1 \preceq \dots \preceq \boldsymbol{s}^m$, $\boldsymbol{L}(\boldsymbol{s}^j) - \boldsymbol{L}(\boldsymbol{s}^{j-1}), j = 2, \dots, m$, are independent,
- for any $\boldsymbol{s}^1 \preceq \boldsymbol{s}^2$ and $\boldsymbol{s}^3 \preceq \boldsymbol{s}^4$ satisfying $\boldsymbol{s}^2 - \boldsymbol{s}^1 = \boldsymbol{s}^4 - \boldsymbol{s}^3$, $\mathcal{L}(\boldsymbol{L}(\boldsymbol{s}^2) - \boldsymbol{L}(\boldsymbol{s}^1)) = \mathcal{L}(\boldsymbol{L}(\boldsymbol{s}^4) - \boldsymbol{L}(\boldsymbol{s}^3))$ where $\mathcal{L}(\cdot)$ denotes the law of the random variable,
- $\boldsymbol{L}(\boldsymbol{0}) = \boldsymbol{0}$ almost surely, and
- almost surely, $\boldsymbol{L}(\boldsymbol{s})$ is right continuous with left limits in $\boldsymbol{s}$ in the partial ordering of $\mathbb{R}^d_+$.

Let $\{\boldsymbol{L}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}^d_+\}$ be a multiparameter Lévy process on $\mathbb{R}^n$ with Lévy triplet $(\boldsymbol{\gamma}_{\boldsymbol{L}}, \Sigma_{\boldsymbol{L}}, \nu_{\boldsymbol{L}})$, and let $\boldsymbol{\tau}(t)$ be a d dimensional subordinator independent of $\boldsymbol{L}(\boldsymbol{s})$ having Lévy triplet $(\boldsymbol{\gamma}_{\boldsymbol{\tau}}, 0, \nu_{\boldsymbol{\tau}})$. The subordinated process $\boldsymbol{X} = \{\boldsymbol{X}(t), t \geq 0\}$ defined by

$$\boldsymbol{X}(t) := \boldsymbol{L}(\boldsymbol{\tau}(t)) = \begin{pmatrix} L_1(\tau_1(t), \dots, \tau_d(t)) \\ \vdots \\ L_n(\tau_1(t), \dots, \tau_d(t)) \end{pmatrix}, \quad t \geq 0$$

is a Lévy process, as proved in Theorem 4.7 in Barndorff-Nielsen et al. (2001), who also provide its characteristic function and Lévy triplet. Our main result provides a link between marked Poisson processes and multivariate subordinated Lévy processes. In particular we give conditions for marks and underlying Poisson measure such that marked Poisson process are in law subordinated Lévy process as defined in Barndorff-Nielsen et al. (2001). In addition we provide their Lévy triplet.

As an example we introduce the class of Gaussian marked Poisson processes and prove that in law they belong to the class of multivariate subordinated Brownian motions. We show that, under suitable conditions, the processes in this class have characteristic functions in closed form. In particular, we focus on a multivariate Laplace process. The Laplace distribution (Laplace, 1774) is infinitely divisible and able to account for heavier than Gaussian tails. For this reason, the multivariate associated Laplace process became popular for multivariate modeling in several areas, as Engineering and Finance (Kotz et al., 2012).

1. Lévy Marked Poisson processes

Here we construct a Marked Poisson process of Lévy type. Let $\boldsymbol{\Pi}$ be a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^d_+, \mathcal{B}_{d+1})$, where $\mathcal{B}_{d+1}$ is the Borel σ algebra, with mean measure $\mu_{\boldsymbol{\Pi}} = Leb \times \nu_{\boldsymbol{\Pi}}$, where $\nu_{\boldsymbol{\Pi}}$ is a Lévy measure, such that $\nu_{\boldsymbol{\Pi}}(\{\boldsymbol{0}\}) = 0$ and $\int_{\mathbb{B}} |\boldsymbol{x}| \nu_{\boldsymbol{\Pi}}(dx) < \infty$, where $\mathbb{B} = \{\boldsymbol{x} \in \mathbb{R}^d, |\boldsymbol{x}| \leq 1\}$ is the unit ball. The process defined by

$$\boldsymbol{\pi}(t) := \int_{(0,t] \times \mathbb{R}^d_+} \boldsymbol{x} \boldsymbol{\Pi}(ds, d\boldsymbol{x}), \tag{1.1}$$

is a zero drift multivariate subordinator with Lévy measure $\nu_{\boldsymbol{\Pi}}$. The atoms of $\boldsymbol{\Pi}$ are family of random variables $\boldsymbol{\Pi} = \{(\Pi_1, \boldsymbol{\Pi}_2) = \{(\Pi_{1i}, \boldsymbol{\Pi}_{2i}),\ i \in I\}\}$ on $\mathbb{R}_+ \times \mathbb{R}^d_+$, where Π_{1i} are the jump times and $\boldsymbol{\Pi}_{2i}$ are the jump sizes.

If $\boldsymbol{L}(\boldsymbol{s})$ is an $\mathbb{R}^d_+$- multiparameter process on $\mathbb{R}^n$ and $\lambda^{\boldsymbol{s}} = \mathcal{L}(\boldsymbol{L}(\boldsymbol{s}))$, and B is any set belonging to $\mathcal{B}_n$ i.e. $B \in \mathcal{B}_n$, we introduce the transition probability kernel Q defined by $Q(\boldsymbol{s}, B) = \lambda^{\boldsymbol{s}}(B)$, i.e.

$$\begin{aligned} Q(\boldsymbol{0}, B) &:= P(\boldsymbol{L}(\boldsymbol{0}) \in B) = \mathbb{1}_B(\boldsymbol{0}) \\ Q(\boldsymbol{s}, B) &:= P(\boldsymbol{L}(\boldsymbol{s}) \in B) \end{aligned} \tag{1.2}$$

and name it *multiparameter Lévy kernel*. The first expression of Eq. (1.2) is a consequence of being $\boldsymbol{L}(\boldsymbol{0}) = \boldsymbol{0}$ with probability one.

The following theorem provides a connection between marked Poisson processes and multivariate subordinated Lévy processes.