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Limiting empirical distribution for eigenvalues of products of random rectangular matrices

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ABSTRACT

We study the empirical spectral distribution of a product $A_N^{(m)} = A_1 \cdots A_m$ of m random rectangular matrices with i.i.d. complex Gaussian entries. The product ensemble is of dimension $N \times N$, and the rectangular matrix A_j is of size $N_j \times N_{j+1}$ for $j = 1, \ldots, m$ with $N_{m+1} = N_1 = N$. Explicit limit of empirical eigenvalue distribution of $A_N^{(m)}$ is obtained in almost sure convergence as N goes to infinity. In particular, a rich feature of the limiting distributions is presented as the ratio N_j/N fluctuates for each j.

1. Introduction

Let m, $\{N_r, r = 1, \ldots, m\}$ be positive integers with $N_{m+1} = N_1 = \min\{N_1, \ldots, N_m\}$. Define for $r = 1, \ldots, m$ the matrix

$$A_r = \begin{pmatrix} g_{11}^{(r)} & g_{12}^{(r)} & \cdots & g_{1N_{r+1}}^{(r)} \\ g_{21}^{(r)} & g_{22}^{(r)} & \cdots & g_{2N_{r+1}}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N_r 1}^{(r)} & g_{N_r 2}^{(r)} & \cdots & g_{N_r N_{r+1}}^{(r)} \end{pmatrix},$$

where $g_{ij}^{(r)}$, $1 \le i \le N_r$, $1 \le j \le N_{r+1}$ are i.i.d. standard complex normal random variables with $\mathbb{E}g_{ij}^{(r)} = 0$, $\mathbb{E}|g_{ij}^{(r)}|^2 = 1$ for $r = 1, \ldots, m$.

For the convenience, we denote $N = N_1$ hereafter. We study the following product of m rectangular matrices given by

$$A_N^{(m)} = A_1 \ldots A_m.$$

The spectral properties of $A_N^{(m)}$ can be applied to wireless telecommunication (Akemann et al., 2013; Muller, 2002; Tulino and Verd, 2004) and transport in disordered and chaotic dynamical system (Crisanti et al., 1993; Ipsen and Kieburg, 2014).

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Particularly for $m = 2$, $A_N^{(2)}$ can be regarded as the asymmetric correlation matrices (see Vinayak, 2013; Vinayak and Benet, 2014) which are widely used in finance (Bouchaud et al., 2007; Bouchaud and Potters, 2009; Livan and Rebecchi, 2012).

If $N_1 = N_2 = \cdots = N_m$, the product matrix $A_N^{(m)}$ is reduced to the product of famous Ginibre ensembles. The exact eigenvalue density first shown by Akemann and Burda (2012) with convergence in the sense of mean value is

$$p(z) = \frac{1}{m\pi}|z|^{\frac{2}{m}-2} I_{\{|z|\le 1\}}. \tag{1.1}$$

Their method is based on a planar diagrammatic technique and Dyson–Schwinger equation. (1.1) can be regarded as the mth power of the eigenvalue density of a single Ginibre ensemble. To be noted, Burda et al. (2012) proved that this kind of result holds for isotropic unitary ensembles. The universality of the limiting spectral measure of the product of independent square random matrices was obtained by Götze and Tikhomirov (2010) in the sense of mean value. Later O'Rourke and Soshnikov (2011) further generalized the universality result in almost sure convergence. And the local law was established by Nemish in Nemish (2015). Very recently, Jiang and Qi (2015) have obtained the limiting spectral distribution of product of arbitrary number (i.e. m can be finite and infinite) of Ginibre ensembles in almost sure convergence.

In the setting N_r/N equals a finite number R_r, $r = 1, \dots, m$, using the same technique from Burda et al. (2010a) for non-Hermitian matrices, it is shown that the M-transforms of product $A_N^{(m)}$ satisfy a mth order polynomial equation, from which the limit eigenvalue density of product can be derived (Burda et al., 2010b).

In this note, we shall focus on a different method to obtain the explicit limiting spectral distribution of $A_N^{(m)}$, and the convergence mode is in almost sure convergence, stronger than the convergence with mean value in Burda et al. (2010b). Besides, we consider a more general case: $N_r/N \to a_r$, as $N \to \infty$, where $a_r \in [1, +\infty]$, $r = 1, \dots, m$.

Set $l_r = N_r - N$, $r = 1, \dots, m$. Let $z_1, \dots, z_N$ be the eigenvalues of $A_N^{(m)}$. The joint density function for the eigenvalues of $A_N^{(m)}$ derived in the Theorem 2 of Adhikari et al. (2013) is

$$p(z_1, \dots, z_N) = C \prod_{1\le j<k\le N} |z_j - z_k|^2 \prod_{j=1}^{N} w_m^{(l_1,\dots,l_m)}(|z_j|) \tag{1.2}$$

with respect to the Lebesgue measure on $\mathbb{C}^N$, where C is a normalizing constant which can be expressed explicitly by Proposition 1 in Jiang and Qi (2015), and $w_m^{(l_1,\dots,l_m)}(z)$ is given in terms of the Meijer G-function as

$$w_m^{(l_1,\dots,l_m)}(z) = (2\pi)^{m-1} G_{0,m}^{m,0}\left[\left.\begin{matrix} - \\ (l_1, l_2, \dots, l_m)\end{matrix}\right| |z|^2\right].$$

By calculation, the weight function $w_m^{(l_1,\dots,l_m)}$ can be expressed as the following recursive formula

$$w_m^{(l_1,\dots,l_m)}(z) = 2\pi \int_0^\infty w_{m-1}^{(l_1,\dots,l_{m-1})}\left(\frac{z}{r}\right) w_1^{(l_m)}(r)\frac{dr}{r} \tag{1.3}$$

with initial $w_1^{(l)}(z) = \exp(-|z|^2)|z|^{2l}$, for any z in the complex plane.

The eigenvalues $z_1, z_2, \dots, z_N$ of the product matrix $A_N^{(m)}$ form a determinant point process, and the k points correlation function is (Adhikari et al., 2013)

$$\rho_k(z_1, \dots, z_k) = \det_{1\le i,j\le k}\left(k(z_i, z_j)\right),$$

with

$$k(z_i, z_j) = \sqrt{w_m(z_i)w_m(\bar{z}_j)} \sum_{r=0}^{N-1} \frac{(z_i\bar{z}_j)^r}{(2\pi)^m \prod_{j=1}^{m}(N_j - N + r)!}$$

and $w_m(z) \equiv w_m^{(l_1,\dots,l_m)}(z)$.

Define

$$\mu_N^{(m)} = \frac{1}{N}\sum_{j=1}^{N} \delta_{z_j/a_N}, \tag{1.4}$$

which is the empirical measure of the eigenvalues $z_1, \dots, z_N$ of $A_N^{(m)}$. From now on, the notation "*Unif* (D)" represents the uniform distribution on set D, and $\mu_1 \otimes \mu_2$ denotes the product measure of two measures μ_1 and μ_2. We use "$\xrightarrow{P}$" for the convergence in probability, and we denote "$\rightsquigarrow$" for the weak convergence of probability measures.