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Weak decreasing stochastic order

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ABSTRACT

We introduce the notion of weak decreasing stochastic (WDS) ordering for real-valued processes with negative means, which, to our knowledge, has not been studied before. Thanks to Madan–Yor's argument, it follows that the WDS ordering is a necessary and sufficient condition for a family of integrable probability measures with negative mean to be embeddable in a standard Brownian motion by the Cox and Hobson extension of the Azéma–Yor algorithm. The resulting process is a supermartingale and if, in addition, the measures have densities, this supermartingale is Markovian. Then the Cox–Hobson algorithm provides a special solution of Kellerer's theorem relying on the stronger hypothesis of WDS order.

1. Introduction

We consider a new stochastic ordering for probability measures with negative means, namely the weak decreasing stochastic (WDS) ordering which is related to the usual stochastic and the increasing convex orders. We recall that a family of probability measures $\mu = (\mu_t, t \in \mathbb{R}_+)$ is said to be non-decreasing in *the usual stochastic order* if, for every $0 \le s \le t$ and every non-decreasing function ϕ such that $\int_{\mathbb{R}} \phi(y)\mu_s(dy)$ and $\int_{\mathbb{R}} \phi(y)\mu_t(dy)$ exist,

$$\int_{\mathbb{R}} \phi(y)\mu_s(dy) \le \int_{\mathbb{R}} \phi(y)\mu_t(dy). \tag{1.1}$$

If (1.1) holds only for non-decreasing convex functions, then μ is said to be non-decreasing in *the increasing convex order*. If μ is non-decreasing in the usual stochastic order, resp. in the increasing convex order, then its image $\mu^h = \left(\mu_t^h, t \ge 0\right)$ under $h : y \longmapsto -y$ is said to be non-decreasing in *the decreasing stochastic order*, resp. in *the decreasing convex order*. We also recall the definition of the MRL ordering which resembles that of the WDS ordering. Suppose that μ_t is integrable for every t. The family μ is said to be non-decreasing in *the MRL order* if the family of functions $\left(\Psi_{\mu_t}^{mrl}, t \in \mathbb{R}_+\right)$ given by

$$\Psi_{\mu_t}^{mrl}(x) = \begin{cases} \dfrac{1}{\mu_t([x, +\infty[)} \displaystyle\int_{[x,+\infty[} y\mu_t(dy) & \text{if } x < r_{\mu_t}, \\ x & \text{otherwise,} \end{cases}$$

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where $r_{\mu_t} = \inf\{z \in \mathbb{R} : \mu_t([z, +\infty[) = 0\}$, is pointwise non-decreasing. Now, we define the WDS ordering as follows. Suppose that, for every $t \geq 0$, μ_t is integrable and has a negative mean. The family μ is said to be non-decreasing in the weak decreasing stochastic (WDS) order if the family of functions $(\Psi^{wds}_{\mu_t}, t \geq 0)$ defined by

$$\Psi^{wds}_{\mu_t}(x) = \begin{cases} \dfrac{1}{\mu_t([x, +\infty[)} \left(\displaystyle\int_{[x,+\infty[} y\mu_t(dy) - m_{\mu_t} \right) & \text{if } x < r_{\mu_t}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $m_{\mu_t} = \int_{\mathbb{R}} y\mu_t(dy)$, is pointwise non-decreasing. A family of integrable real-valued random variables with negative means is said to be non-decreasing in the WDS order if the family of their respective distributions is non-decreasing in the WDS order. Observe that the definition of the WDS ordering is the same as that of the MRL ordering up to the subtraction of the mean of μ_t and the value of $\Psi^{wds}_{\mu_t}$ if $x \geq r_{\mu_t}$. The terminology *weak decreasing stochastic ordering* has been chosen since, for processes with negative mean, the usual decreasing stochastic order is strictly stronger than the WDS order. Indeed, we show that every stochastically non-increasing process with negative mean is ordered by the WDS order and we exhibit some WDS ordered processes which do not decrease stochastically. On the other hand, we prove that, for processes with negative mean, the WDS order strictly implies the decreasing convex order. In particular, WDS ordered processes with constant negative mean are necessarily stochastically constant. Such a result has been proved by Shaked and Shanthikumar (2007, Theorem 1.A.8) for stochastically non-increasing processes. One may also define a notion of *weak increasing stochastic (WIS) ordering* for processes with positive means. A family of integrable probability measures $\nu = (\nu_t, t \geq 0)$ with positive means is said to be non-decreasing in *the WIS order* if the family of functions $\left(\Psi^{wis}_{\nu_t}, t \geq 0\right)$ given by

$$\forall\, t \geq 0, \quad \Psi^{wis}_{\nu_t}(x) = \begin{cases} \dfrac{1}{\nu_t(]-\infty, x])} \left(m_{\nu_t} - \displaystyle\int_{]-\infty,x]} y\nu_t(dy) \right) & \text{if } x > l_{\nu_t} \\ +\infty & \text{otherwise,} \end{cases}$$

where $m_{\nu_t} = \int y\nu_t(dy)$ and $l_{\nu_t} = \sup\{z \in \mathbb{R} : \nu_t(]-\infty, z]) = 0\}$. Observe that, if ν^h_t denotes the image of ν_t under h, then, for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, $\Psi^{wis}_{\nu_t}(x) = \Psi^{wds}_{\nu^h_t}(-x)$. This implies that $\nu = (\nu_t, t \geq 0)$ is non-decreasing in the WIS order if, and only if $\nu^h = \left(\nu^h_t, t \geq 0\right)$ is non-decreasing in the WDS order. As a consequence, the WIS ordering is strictly weaker than the usual stochastic ordering and strictly stronger than the increasing convex ordering. Recently, Ewald and Yor (2015, Definition 1) introduced the notion of *a lyrebird* and called lyrebird a process that is non-decreasing in the increasing convex order. Hence the class of lyrebirds includes strictly that of WIS ordered processes.

There is a connection between the WDS ordering and the Cox–Hobson embedding. Indeed, the Cox–Hobson stopping time T_{μ_t} that solves the Skorokhod embedding problem[1] for μ_t is the first time the process $\left(B_v, S_v := \sup_{0\leq w\leq v} B_v; v \geq 0\right)$ hits the epigraph $\mathcal{E}^{wds}_{\mu_t}$ of $\Psi^{wds}_{\mu_t}$, where $(B_v, v \geq 0)$ denotes a standard Brownian motion issued from 0. Hence the family $(\mu_t, t \geq 0)$ is non-decreasing in the WDS order if, and only if $(\mu_t, t \geq 0)$ can be embedded in a standard Brownian motion meaning that $t \longmapsto T_{\mu_t}$ is a.s. non-decreasing. Since each T_{μ_t} is minimal (in a sense that is made more precised in the sequel), the process $\left(B_{T_{\mu_t}}, t \geq 0\right)$ is a supermartingale with the same one-dimensional marginals as $(\mu_t, t \geq 0)$. Then it follows from the Jensen inequality that the WDS order is stronger than the decreasing convex order. We recover this property using a different approach. Note that $\left(B_{T_{\mu_t}}, t \geq 0\right)$ is Markovian when the distributions $\mu_t, t \in \mathbb{R}_+$ have densities. This follows from a similar argument than that used in Madan and Yor (2002, Theorem 2). If there is some distribution μ_s with atoms, then the atomic part of μ_s makes some parts of $\mathcal{E}^{wds}_{\mu_s}$ vertical. Thus, for $s < t$, the future random time T_{μ_t} does not only depend on $B_{T_{\mu_s}}$ but also on $S_{T_{\mu_s}}$. Then $\left(B_{T_{\mu_t}}, t \geq 0\right)$ is not Markovian. We say that two processes are associated if they have the same one-dimensional marginals. Let us mention that the problem of existence of a supermartingale associated to a given process which is ordered by the decreasing convex order was solved by Kellerer (1972).

In this paper, we provide a log-concavity characterization of the WDS ordering. This characterization is the same as that obtained in Bogso (2015, Theorem 3.3) for the MRL ordering.

We organize the rest of the paper as follows. In the next section, we briefly recall the Cox and Hobson (2006) extension of the Azéma–Yor algorithm to target distributions with negative mean and, using a Madan–Yor argument, deduce that WSD ordering is a necessary and sufficient condition for an integrable process with negative mean to embeddable in Brownian motion by the generalized Azéma–Yor stopping times. Section 3 is devoted to a characterization of the WDS ordering in terms of log-concavity and to some of its closure properties. Finally, in Section 4, we present several examples of WDS ordered processes.

2. WDS order and the Cox–Hobson algorithm

Let $(\mu_t, t \geq 0)$ be a family of integrable probability measures with negative mean. For every $t \geq 0$, we set $m_{\mu_t} := \int y\mu_t(dy)$. We shall apply the Cox and Hobson (2006) extension of the Azéma–Yor algorithm to embed simultaneously all

[1] The Skorokhod embedding problem, which was first stated and solved by Skorokhod (1965), may be described as follows: Given a Brownian motion $(B_v, v \geq 0)$ and a centered target law μ, does there exist a stopping time T such that B_T has distribution μ?