Statistics and Probability Letters 126 (2017) 65–75

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Large deviations for estimators of the parameters of a neuronal response latency model

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ARTICLE INFO

Article history:
Received 20 April 2016
Accepted 20 February 2017
Available online 1 March 2017

MSC:
60F10
62F12
62F15
92C20

Keywords:
Contraction principle
Relative entropy
Maximum likelihood estimator
Curved exponential model

ABSTRACT

We consider a model in the literature for the neuronal activity with response latency. We present large deviation results for two sequences of estimators of some unknown parameters. We also present a large deviation result for the posterior distributions in the Bayesian setting.

1. Introduction

In this paper we consider the neuronal response latency model in Tamborrino et al. (2012, 2013) (see Definition 1.1 for a rigorous presentation of the statistical model). More precisely we are interested in two sequences of estimators (the maximum likelihood estimators, and a hybrid estimator in the above references) and the sequence of posterior distributions in the Bayesian setting. Our results concern the asymptotic behavior of these sequences in the fashion of large deviations (see e.g. Dembo and Zeitouni, 1998 as a reference on this topic) as the sample size goes to infinity.

We briefly recall the neural activity framework. At time zero, the measurements start and spikes due to the spontaneous activity are recorded. At time t_s, a stimulus is applied and the measurements are stopped after the first spike following t_s, and we denote the time elapsed since t_s by T. The experiment is repeated n times, after a period of time long enough to ensure that the effect of stimulation has disappeared. This allows to obtain n i.i.d. random variables $T_1, \ldots, T_n$. The spontaneous activity does not disappear after the stimulus onset and therefore the first spike T after t_s is not necessarily evoked. Moreover, an observer cannot distinguish whether T is due to spontaneous or evoked activity. Then, for the ith experiment, we have $T_i = \min\{\theta + Z_i, W_i\}$ where $T_i = W_i$ if the first spike is due to the spontaneous activity, and $T_i = \theta + Z_i$ if the first spike is due to the stimulus; moreover θ is a (known) delay parameter, and Z_i and W_i are independent exponentially distributed random variables with means $1/\omega$ and $1/\lambda$, respectively. We also have n i.i.d. random variables $X_1, \ldots, X_n$ (independent of $T_1, \ldots, T_n$), where X_i counts the number of spontaneous spikes occurring in the time interval $[0, t_s]$ before the ith stimulation (see Section 2.1 in Tamborrino et al., 2013); we assume that $t_s = 1$ (for simplicity) and $X_1, \ldots, X_n$ are Poisson distributed

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http://dx.doi.org/10.1016/j.spl.2017.02.026

with mean λ (this assumption is commonly assumed and, as pointed out in Gerstner and Kistler (2002) and Tuckwell (1988), in some cases it is supported by the experimental data).

Now we recall the rigorous definition of the statistical model. Note that, throughout this paper, we use capital letters to stress that the quantities are random variables and lowercase letters to indicate their realizations; moreover we use the symbol $\otimes$ for the product measure, i.e. for the joint distribution of independent random variables having suitable marginal distributions.

Definition 1.1. Let $(\omega, \lambda) \in (0, \infty)^2$ be arbitrarily fixed, and let $P_{\omega,\lambda}^{\otimes n}$ be the joint distribution of the following random variables $(X_1, T_1), \dots, (X_n, T_n)$. We assume that $\{X_1, \dots, X_n\}$ and $\{T_1, \dots, T_n\}$ are independent, $\{X_1, \dots, X_n\}$ are i.i.d. Poisson distributed with mean λ, and $\{T_1, \dots, T_n\}$ are i.i.d. with (common) distribution function

$$F_T(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-\lambda t} & \text{if } 0 \le t < \theta \\ 1 - e^{-\lambda t} + e^{-\lambda t}(1 - e^{-\omega(t-\theta)}) & \text{if } t \ge \theta \end{cases}$$

for some $\theta > 0$ (see e.g. (7) in Tamborrino et al., 2012, or (3) in Tamborrino et al., 2013, with $F_W(t) = (1 - e^{-\lambda t})1_{t>0}$ and $F_Z(t) = (1 - e^{-\omega t})1_{t>0}$). Therefore the random variables $\{T_1, \dots, T_n\}$ have continuous density

$$f_T(t) = \begin{cases} 0 & \text{if } t \le 0 \\ \lambda e^{-\lambda t} & \text{if } 0 < t < \theta \\ (\lambda + \omega)e^{-(\lambda+\omega)t}e^{\omega\theta} & \text{if } t \ge \theta \end{cases} \tag{1}$$

(see e.g. (23) in Tamborrino et al., 2012, or (11) in Tamborrino et al., 2013). For the inference problem we assume that (ω, λ) is unknown and θ is known.

The log-likelihood concerning the n-sample $(x_1, t_1), \dots, (x_n, t_n)$ is

$$\sum_{i=1}^n \log\left\{\frac{\lambda^{x_i}}{x_i!}e^{-\lambda}\right\} + \sum_{i=1}^n \log f_T(t_i) = \sum_{i=1}^n x_i \log\lambda - n\lambda - \sum_{i=1}^n \log(x_i!) + \sum_{i=1}^n \log f_T(t_i)$$

and, if we neglect the constant term $\sum_{i=1}^n \log(x_i!)$, we can restrict the attention to

$$\ell_n(\omega, \lambda) = \sum_{i=1}^n x_i \log\lambda - n\lambda + \sum_{i=1}^n \log f_T(t_i).$$

Moreover, as far as the last sum is concerned, for $t_i > 0$ we have

$$\begin{aligned} \log f_T(t_i) &= \begin{cases} \log\lambda - \lambda t_i & \text{if } 0 < t_i \le \theta \\ \log(\lambda+\omega) - (\lambda+\omega)t_i + \omega\theta & \text{if } t_i > \theta \end{cases} \\ &= -\lambda t_i + \begin{cases} \log\lambda & \text{if } 0 < t_i \le \theta \\ \log(\lambda+\omega) - \omega(t_i - \theta) & \text{if } t_i > \theta; \end{cases} \end{aligned}$$

then

$$\ell_n(\omega, \lambda) = n(\bar{x}_n \log\lambda - \lambda - \lambda\bar{t}_n + (1 - \bar{y}_n)\log\lambda + \bar{y}_n \log(\lambda+\omega) - \omega\bar{u}_n),$$

where

$$\bar{x}_n := \frac{1}{n}\sum_{i=1}^n x_i, \qquad \bar{t}_n := \frac{1}{n}\sum_{i=1}^n t_i, \qquad \bar{y}_n := \frac{1}{n}\sum_{i=1}^n 1_{t_i>\theta} \quad \text{and} \quad \bar{u}_n := \frac{1}{n}\sum_{i=1}^n (t_i - \theta)1_{t_i>\theta}. \tag{2}$$

Remark 1.1. We have a curved exponential model. In fact

$$\frac{\ell_n(\omega, \lambda)}{n} = \sum_{i=1}^4 s_n(i)\eta_i(\omega, \lambda) - \Psi(\eta_1(\omega, \lambda), \eta_2(\omega, \lambda), \eta_3(\omega, \lambda), \eta_4(\omega, \lambda)),$$

where

$$\begin{cases} \eta_1(\omega, \lambda) = \lambda \\ \eta_2(\omega, \lambda) = \omega \\ \eta_3(\omega, \lambda) = \log(\lambda+\omega) \\ \eta_4(\omega, \lambda) = \log\lambda, \end{cases} \quad \begin{cases} s_n(1) = -\bar{t}_n \\ s_n(2) = -\bar{u}_n \\ s_n(3) = \bar{y}_n \\ s_n(4) = \bar{x}_n - \bar{y}_n \end{cases} \quad \text{and} \quad \Psi(\eta_1, \eta_2, \eta_3, \eta_4) = \eta_1 - \eta_4.$$