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On inequalities for values of first jumps of distribution functions and Hölder's inequality

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ABSTRACT

We derive moments inequalities for values of jumps of distribution functions at the infimum points for bounded discrete random variables. We discuss relationships of these inequalities with bounds for probabilities of unions of events and the Cauchy–Bunyakovski and Hölder inequalities.

1. Introduction

Inequalities for probabilities of unions of events are widely used in probability theory and its applications. Such results have been obtained in Chung and Erdős (1952), Gallot (1966), Dawson and Sankoff (1967), Kounias (1968), Kwerel (1975a,b,c), Móri and Székely (1985), Boros and Prékopa (1989), Kounias and Sotirakoglou (1993), Galambos and Simonelli (1996), de Caen (1997), Kuai et al. (2000), Feng et al. (2009), Prékopa (2009), Frolov (2012, 2014, 2015a,b) and references therein. (See also references on the Borel–Cantelli lemma.) In there, various methods have been applied to derive these inequalities. One of them was developed in Frolov (2012, 2015a). In this paper, we apply this method to bound a value of jump at zero for the distribution function of a non-negative discrete random variable. Moreover, we discuss relationships of these inequalities with bounds for probabilities of unions of events and the Cauchy–Bunyakovski and Hölder inequalities.

We first describe ideas which our approach is based on. To this end, we consider two simple examples.

Let $A_1, A_2, \dots, A_n$ be events. Put $U_n = A_1 \cup A_2 \cup \dots \cup A_n$ and

$$\xi_n = \sum_{i=1}^{n} I_{A_i},$$

where I_{A_i} is the indicator of the event A_i, $i = 1, 2, \dots, n$. Put $p_i = \mathbf{P}(\xi_n = i)$ for $i = 0, 1, 2, \dots, n$. Simplest bounds for $\mathbf{P}(U_n)$ are based on two first moments of ξ_n:

$$\mu_1 = \mathbf{E}\xi_n = \sum_{i=1}^{n} i p_i, \qquad \mu_2 = \mathbf{E}\xi_n^2 = \sum_{i=1}^{n} i^2 p_i.$$

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Let m be a fixed natural number such that $2 \leqslant m \leqslant n$. Then the following inequality holds:

$$0 \leqslant \sum_{i=1}^{n} \left(1 - \frac{i}{m-1}\right)\left(1 - \frac{i}{m}\right) p_i = \sum_{i=1}^{n} p_i - \frac{2m-1}{m(m-1)}\mu_1 + \frac{1}{m(m-1)}\mu_2. \tag{1}$$

Hence,

$$\mathbf{P}(U_n) = 1 - p_0 = \sum_{i=1}^{n} p_i \geqslant \frac{2m-1}{m(m-1)}\mu_1 - \frac{1}{m(m-1)}\mu_2. \tag{2}$$

Since this inequality is satisfied for every m, we can optimize it by varying m. It is clear that the inequality in (1) turns to equality for distributions of ξ_n concentrated in the points $0, m-1$ and m. Choose such distribution with two first moments equal to μ_1 and μ_2, correspondingly. To this goal, we solve the following linear system:

$$(m-1)p^*_{m-1} + mp^*_m = \mu_1,$$
$$(m-1)^2 p^*_{m-1} + m^2 p^*_m = \mu_2.$$

We get

$$p^*_{m-1} = \frac{m\mu_1 - \mu_2}{m-1}, \qquad p^*_m = \frac{\mu_2 - (m-1)\mu_1}{m}.$$

The conditions $p^*_{m-1} \geqslant 0$ and $p^*_m \geqslant 0$ yield $\mu_2/\mu_1 \leqslant m \leqslant \mu_2/\mu_1 + 1$. Substituting the expression $m = \mu_2/\mu_1 - \theta + 1$ in (2), we arrive at the Dawson–Sankoff inequality:

$$\mathbf{P}(U_n) \geqslant \frac{\theta\mu_1^2}{\mu_2 + (1-\theta)\mu_1} + \frac{(1-\theta)\mu_1^2}{\mu_2 - \theta\mu_1},$$

where $\theta = \mu_2/\mu_1 - [\mu_2/\mu_1]$ and $[\cdot]$ is the integer part of the number in brackets. Note that θ may be positive. The right-hand side of the Dawson–Sankoff inequality is minimal for $\theta = 0$. It yields the following well known Chung–Erdős inequality:

$$\mathbf{P}(U_n) \geqslant \frac{\mu_1^2}{\mu_2}.$$

The above method yielded an upper bound for p_0 that is the value of the first jump (jump at zero) of the distribution function of ξ_n. Instead of ξ_n, consider the random variable η_n such that $q_i = \mathbf{P}(\eta_n = x_i)$ for $i = 1, 2, \ldots, n$, and $0 < x_1 < x_2 < \cdots < x_n$. Put

$$s_1 = \mathbf{E}\eta_n = \sum_{i=1}^{n} x_i q_i, \qquad s_2 = \mathbf{E}\eta_n^2 = \sum_{i=1}^{n} x_i^2 q_i.$$

An analogue of (1) is

$$0 \leqslant \sum_{i=1}^{n} \left(1 - \frac{x_i}{x_{m-1}}\right)\left(1 - \frac{x_i}{x_m}\right) q_i = 1 - \frac{x_m + x_{m-1}}{x_m x_{m-1}} s_1 + \frac{1}{x_m x_{m-1}} s_2. \tag{3}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ be positive real numbers such that

$$\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \beta_i^2 = 1.$$

Put $q_i = \beta_i^2$ and $x_i = \alpha_i/\beta_i$ for $i = 1, 2, \ldots, n$. (We assume now for simplicity that all x_i are different. We deal with the general case below.) Then (3) turns to

$$0 \leqslant 1 - \frac{x_m + x_{m-1}}{x_m x_{m-1}} \sum_{i=1}^{n} \alpha_i\beta_i + \frac{1}{x_m x_{m-1}}.$$

In the same way as before, we get an upper bound for $\sum_{i=1}^{n} \alpha_i\beta_i$. For the Chung–Erdős inequality, we then have the following analogue:

$$\sum_{i=1}^{n} \alpha_i\beta_i \leqslant 1.$$

This is the Cauchy–Bunyakovski inequality.