Statistics and Probability Letters 126 (2017) 157–163

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Asymptotic distributions of some robust scale estimators in explosive AR(1) model

Hira L. Koul[a], P. Vellaisamy[b,*]

[a] Department of Statistics & Probability, Michigan State University, East Lansing, MI 48824-1027, USA
[b] Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai-400076, India

ARTICLE INFO

Article history:
Received 26 January 2017
Received in revised form 2 March 2017
Accepted 7 March 2017
Available online 20 March 2017

MSC:
primary 62G05
secondary 62G35

Keywords:
Median
Absolute residuals
Pairwise absolute differences

ABSTRACT

This note establishes the asymptotic normality of the median of the absolute residuals and the median of the absolute differences of pairwise residuals in the first order explosive autoregressive time series. These estimators are useful for obtaining some scale invariant estimators of the autoregressive parameter.

1. Introduction

In this note, we derive the limiting distributions of the two robust scale estimators in explosive autoregressive time series of order one. Proceeding a bit more precisely, let $X_i, i \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, be an autoregressive time series of order 1, i.e., for some $\rho \in \mathbb{R} := (-\infty, \infty)$,

$$X_i = \rho X_{i-1} + \varepsilon_i, \quad i \in \mathbb{Z}, \tag{1.1}$$

where the errors $\varepsilon_i, i \in \mathbb{Z}$, are independent and identically (i.i.d.) random variables (r.v.'s), and ε_i is independent of X_{i-1}, for each $i \in \mathbb{Z}$. It is well known that the asymptotic behavior of the various inference procedures for ρ is affected by the values of ρ, i.e., whether $|\rho| < 1$, or $\rho = 1$, or $|\rho| > 1$. In the case $|\rho| > 1$, the model (1.1) is called the explosive AR(1) time series. Its importance is discussed in Anderson (1959) and White (1958).

Now, let $\psi(y), y \in \mathbb{R}$, be a real valued function such that $E\psi(\varepsilon) = 0$ and let

$$M(\boldsymbol{X}, t) := \sum_{i=1}^{n} X_{i-1}\psi(X_i - tX_{i-1}), \quad t \in \mathbb{R}, \tag{1.2}$$

where $\boldsymbol{X} := (X_0, X_1, \ldots, X_n)'$. Analogous to the regression setup of Huber (1981), an M-estimator $\widehat{\rho}(\boldsymbol{X})$ of ρ, based on $\boldsymbol{X}$ and corresponding to the given ψ, is defined as a solution t of the equation $M(\boldsymbol{X}, t) = 0$. Numerous authors have studied these estimators under a variety of conditions. See, for example, Bustos (1982), Davis et al. (1992), Koul (2002a,b), and the

* Corresponding author.
E-mail addresses: koul@stt.msu.edu (H.L. Koul), pv@math.iitb.ac.in (P. Vellaisamy).

http://dx.doi.org/10.1016/j.spl.2017.03.007

references therein. Unlike in linear regression setup, these estimators are not robust against heavy tail error distributions. Denby and Martin (1979) studied a generalized versions of these estimators that have this robustness property.

An estimator $\widetilde{\rho}(\boldsymbol{X})$ of ρ is said to be scale invariant if for any $a > 0$, $\widetilde{\rho}(a\boldsymbol{X}) \equiv \widetilde{\rho}(\boldsymbol{X})$. From a practical point of view, it is desirable to have scale invariant estimators of ρ. The above M-estimators, obtained from (1.2), are not scale invariant. An exception is the least squares estimator, which corresponds to the case $\psi(y) \equiv y$. A way to make general M-estimators scale invariant and robust at the same time is to modify the above definition as follows. Let $s(\boldsymbol{X})$ be a scale parameter estimator having the property that $s(\boldsymbol{X}) > 0$ and for any $a > 0$, $s(a\boldsymbol{X}) = a\,s(\boldsymbol{X})$. Let g be a real valued bounded function, $\boldsymbol{X}_{i-1} := (X_0, X_1, \ldots, X_{i-1})'$ and $h(\boldsymbol{X}_{i-1})$ be a positive function of $\boldsymbol{X}_{i-1}$ such that $h(a\boldsymbol{X}_{i-1}) = ah(\boldsymbol{X}_{i-1})$ for all $a > 0$ and all $1 \le i \le n$. Let

$$\widehat{M}(\boldsymbol{X}, t) := \sum_{i=1}^{n} g\Big(\frac{X_{i-1}}{h(\boldsymbol{X}_{i-1})}\Big)\psi\Big(\frac{X_i - tX_{i-1}}{s(\boldsymbol{X})}\Big).$$

Define $\widehat{\rho}(\boldsymbol{X})$ as a solution t of the equation

$$\widehat{M}(\boldsymbol{X}, t) = 0. \tag{1.3}$$

Note that for any $a > 0$, $\widehat{M}(a\boldsymbol{X}, t) \equiv \widehat{M}(\boldsymbol{X}, t)$ and $\widehat{\rho}(a\boldsymbol{X}) = \widehat{\rho}(\boldsymbol{X})$, w.p.1. An example of g and h one may use is $g(y) \equiv y, y \in \mathbb{R}$, and $h(\boldsymbol{X}_{i-1}) := \sqrt{\sum_{j=1}^{i} X_{j-1}^2}$.

Let $\widetilde{\rho}$ be a invariant estimator of ρ and $r_i := X_i - \widetilde{\rho}X_{i-1}$ and $\bar{r} := n^{-1}\sum_{i=1}^{n} r_i$. Some examples of the scale invariant scale estimators of scale parameters are

$$s_0 := \Big\{n^{-1}\sum_{i=1}^{n} r_i^2\Big\}^{1/2}, \qquad s_1 := \text{median}\{|X_j - \tilde{\rho}X_{j-1}|,\ 1 \le j \le n\},$$
$$s_2 := \text{median}\{|(X_j - \tilde{\rho}X_{j-1}) - (X_i - \tilde{\rho}X_{i-1})|,\ 1 \le i < j \le n\}.$$

Let $\sigma^2 := E\varepsilon^2$ whenever $E\varepsilon^2 < \infty$. Let σ_1 and σ_2 denote median of the distribution of $|\varepsilon_1|$ and $|\varepsilon_1 - \varepsilon_2|$, respectively. Note that s_0^2, s_1 and s_2 are scale invariant estimators of σ^2, σ_1 and σ_2, respectively. It is well known that s_0 is sensitive to the heavy tail error distribution, while s_1, s_2 are robust against such error tails, cf., Davis et al. (1992).

The estimators $\widehat{\rho}$ defined in (1.3) are *a priori* scale invariant and robust against heavy tail error distributions, cf., Denby and Martin (1979). In order to retain the robustness property of $\widehat{\rho}$, it is desirable to take $s(\boldsymbol{X})$ in (1.3) to be equal to the robust scale estimators s_1 or s_2. To establish the asymptotic properties of the corresponding M-estimators $\widehat{\rho}(\boldsymbol{X})$, we need s_1 and s_2 to be at least consistent. It is also desirable to know the asymptotic distributions of their suitably standardized versions.

Koul (2002a,b) investigated the asymptotic distributions of the analogs of s_1 and s_2 for a class of general nonlinear regression models, when errors are i.i.d. having finite variance. In particular, some results of this paper imply that in the stationary autoregressive model (1.1) with $|\rho| < 1$ and $E(\varepsilon) = 0, E\varepsilon^2 < \infty$, the asymptotic distributions of $\delta_j := n^{1/2}(s_j - \sigma_j)/\sigma_j, j = 1, 2$, are not affected by not knowing ρ. In other words, if we let $\tau_1 := \text{median}\{|\varepsilon_i|, 1 \le i \le n\}$, $\tau_2 := \text{median}\{|\varepsilon_j - \varepsilon_i|, 1 \le i < j \le n\}$, $\tilde{\delta}_j := n^{1/2}(\tau_j - \sigma_j)/\sigma_j, j = 1, 2$, then the asymptotic distribution of δ_j is the same as that of $\tilde{\delta}_j, j = 1, 2$. This is unlike what happens in the linear regression setup, where the asymptotic distribution of $n^{1/2}(s_1 - \sigma_1)/\sigma_1$ is seriously influenced by the estimators of the regression parameters, unless the errors are symmetrically distributed around the origin or the design variables are centered at zero.

In the current paper, we derive the asymptotic distributions of s_1 and s_2 in the case of explosive autoregressive model of order 1, where $|\rho| > 1$. The main findings of this note are similar to those in the non-explosive case where $|\rho| < 1$, i.e., the asymptotic distributions of these two estimators are not affected by not knowing ρ. In the case $|\rho| < 1$, we had to assume that the errors have finite variance while in the explosive case these results are valid under relatively a weaker assumption requiring only $E\log(1 \vee |\varepsilon|) < \infty$.

2. Main results

This section describes the main results about the asymptotic distributions of suitably standardized s_1 and s_2, under the explosive autoregressive setup. Accordingly, consider the model (1.1) with $X_0 = 0$ and $|\rho| > 1$. Let ε_i's be i.i.d. F so that

$$p_1(y) := F(y) - F(-y), \qquad p_2(y) := \int [F(y + x) - F(-y + x)]dF(x), \quad y \ge 0$$

denote the distribution functions of $|\varepsilon_1|$ and $|\varepsilon_1 - \varepsilon_2|$, respectively.

Define σ_1 and σ_2 by the relations

$$p_1(\sigma_1) = 1/2, \qquad p_2(\sigma_2) = 1/2. \tag{2.1}$$

Clearly, σ_1 (σ_2) is a median of the distribution of $|\varepsilon_1|$ ($|\varepsilon_1 - \varepsilon_2|$).