Statistics and Probability Letters 126 (2017) 179–184

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Error bounds for kernel density estimator of spectral distribution for Gaussian Unitary Ensembles

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ARTICLE INFO

Article history:
Received 14 December 2016
Received in revised form 11 March 2017
Accepted 14 March 2017
Available online 21 March 2017

Keywords:
Kernel density estimator
Gaussian unitary ensembles
Deviation inequality
L_1 error bound

ABSTRACT

In this short note, we study the error bounds for kernel density estimator of spectral distribution for Gaussian unitary ensembles. A deviation inequality and L_1 error bound for the kernel estimator have been obtained. Our approach is based on an abstract deviation inequality for linear spectrum statistics of determinantal points fields.

1. Introduction and main results

1.1. Introduction

The study of random matrix theory traces back to 1928, when Wishart (1928) considered the Fisher classification problem of multivariate samples. By the pioneer work of Wigner (1955, 1958) and his coauthors in studying the quantum mechanics in the 1940s and early 1950s, random matrix theory becomes an important subject. In the late 1950s, researches on the limiting spectral analysis of large dimensional random matrices have attracted considerable interest and has developed significantly during the past few decades, no matter in theories or applications, see for example Bai and Silverstein (2006) and Mehta (1990).

In this paper, we consider the Gaussian Unitary Ensemble (GUE) defined below. Let M_n be an $n \times n$ random Hermitian matrix of the form

$$W_n = \frac{1}{\sqrt{n}} M_n = \frac{1}{\sqrt{n}} \{\xi_{jk}\}_{j,k=1}^{n},$$

in which $\xi_{ll} \in \mathbb{R}$, $1 \le l \le n$, $\xi_{jk} = \bar{\xi}_{kj} \in \mathbb{C}$, $1 \le j < k \le n$, and $\{\xi_{ll}, \xi_{jk}; 1 \le i \le n, 1 \le j < k \le n\}$ is a collection of independent variables such that $\mathbb{E}\xi_{ll} = \mathbb{E}\xi_{jk} = 0$, $\mathbb{E}|\xi_{jk}|^2 = 1$ and $\mathbb{E}\xi_{ll}^2 = \sigma^2 < \infty$. Moreover, if the entries are Gaussian distributed, i.e.

$$\xi_{ll} \sim N(0,1)_{\mathbb{R}}, \quad 1 \le l \le n, \qquad \xi_{jk} \sim N(0,1)_{\mathbb{C}}, \quad 1 \le j < k \le n.$$

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http://dx.doi.org/10.1016/j.spl.2017.03.014
0167-7152/

Here $N(0,1)_{\mathbb{R}}$ (resp. $N(0,1)_{\mathbb{C}}$) represents the standard real (resp. complex) Gaussian distribution, we say W_n is Gaussian Unitary Ensemble (GUE).

For GUE matrix M_n, we denote its ordered eigenvalues as $\lambda_1(W_n) \leq \lambda_2(W_n) \leq \cdots \leq \lambda_n(W_n)$. The famous semi-circle law (Wigner, 1955, 1958) says that the empirical spectral distribution (ESD) of W_n which is defined by

$$F^{W_n}(x) =: \frac{1}{n}\sum_{i=1}^{n} I(\lambda_i(W_n) \leq x) \tag{1.1}$$

converges weakly to semi-circle law ρ_{sc}, where $I(\cdot)$ is the indicator function and

$$\rho_{sc}(\mathrm{d}x) = \frac{1}{2\pi^2}\sqrt{4-x^2}\mathrm{d}x, \quad -2 \leq x \leq 2.$$

That means for any bounded and continuous function f,

$$\frac{1}{n}\sum_{i=1}^{n} f(\lambda_i(W_n)) \to \int f(x)\rho_{sc}\mathrm{d}x$$

in probability as $n \to \infty$. Above convergence issue can be proved under more weaker conditions, for example, we only need assumptions about the moments of underlying distributions. One can refer to Anderson et al. (2010) and Bai and Silverstein (2006) as standard references for random matrix theory.

Let $F(x)$ be the cumulative distribution function of ρ_{sc}. In statistical inferences aspects, it is natural to ask how to estimate ρ_{sc} and F. The first natural candidate should be F^{W_n}. But unfortunately, there is no central limit theorem for $F^{W_n} - F$ (cf. Bai and Silverstein, 2006), this will cause difficulties in constructing confidence intervals. More recently, Jing et al. (2010), Pan et al. (0000) and Zhou (2013) proposed the kernel density estimator of F^{W_n} for sample covariance matrices and Wigner matrices, respectively. Under some moments assumptions, Zhou (2013) proved that the kernel density estimator is strong consistent in the uniform topology. In this note, we will give a deviation inequality and L_1 error bound for the kernel density estimator based on a connection with linear spectrum statistics of orthogonal polynomial ensembles in the mesoscopic scale (Breuer and Duits, 2014).

To fix the main idea, we give some illustrations of kernel density estimator. Suppose that the observations of $X_1, \ldots, X_n$ are i.i.d. random variables with an unknown density function $g(x)$, distribution function F and denote $F_n(x)$ is the empirical distribution function determined by the samples. Then a popular nonparametric estimate of $g(x)$ is

$$\hat{f}_n(x) = \frac{1}{nh}\sum_{j=1}^{n} K\left(\frac{x-X_j}{h}\right) = \frac{1}{h}\int K\left(\frac{x-y}{h}\right)\mathrm{d}F_n(y), \tag{1.2}$$

where $K(\cdot)$ is known as the kernel function, which is a smooth function such that

$$K(x) \geq 0, \quad \int K(x)\mathrm{d}x = 1. \tag{1.3}$$

Under some regular conditions on the kernel, it is well known that $\hat{f}_n(x) \to g(x)$ in some sense, cf. Hall (1984), Parzen (1962), Rosenblatt (1956) and Silverman (1986). Also, it is natural to use

$$\hat{F}_n(x) = \int_{-\infty}^{x} \hat{f}_n(u)\mathrm{d}u \tag{1.4}$$

to estimate F. Using this idea, in this paper, we consider the kernel density estimation for ρ_{sc} and F^{W_n}, here W_n is a GUE matrix and F^{W_n} is given by (1.1). In the same spirit, we can define

$$f_n(x) = \frac{1}{h}\int K\left(\frac{x-y}{h}\right)\mathrm{d}F^{W_n}(y), \tag{1.5}$$

and

$$F_n(x) = \int_{-\infty}^{x} f_n(u)\mathrm{d}u. \tag{1.6}$$

1.2. Main results

Our main results are following deviation inequality for $f_n(x)$ and L_1 error bound for $F_n(x)$.

Theorem 1.1. *Let the bounded kernel function K satisfies (1.3). Assume that the bandwidth $h := h(n) \to 0$ and $nh^2 \to \infty$. Then*