Statistics and Probability Letters 126 (2017) 212–218

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Continuum percolation with holes

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ARTICLE INFO

Article history:
Received 16 August 2016
Received in revised form 27 December 2016
Accepted 8 March 2017
Available online 22 March 2017

Keywords:
Percolation
Poisson process
Gilbert model
Cognitive networks
Simultaneous connectivity

ABSTRACT

We analyze a mathematical model of a cognitive radio network introduced in Yemini et al. (2016). Our analysis reveals several surprising features of the model. We explain some of these features using ideas from percolation theory and stochastic geometry.

1. Introduction

Percolation on the standard disc graph (Gilbert's disc model) has been a well-studied topic since the seminal work of Gilbert (1961). It has applications in wireless ad hoc or sensor networks (Haenggi, 2012), where it is assumed that the network is composed of a single class of transceivers with a fixed transmission radius. In an important emerging class of networks, the so-called cognitive networks, however, there exist two classes of transceivers, where the so-called secondary users are only allowed to be active if they are not too close to any of the primary users (Lee and Haenggi, 2012). In these networks, the primary users are allowed unrestricted access to their licensed radio spectrum, while the secondary users are prohibited from causing harmful interference to the primary users, i.e., they need to respect a guard zone around the primary users.

We focus on percolation in the network formed by the secondary users. Assuming that primary and secondary users form independent Poisson point processes, the subset of secondary users who are allowed to be active is a *Poisson hole process*, since the guard zones around the primary users create holes in the point process of active secondary users. This point process was introduced in Lee and Haenggi (2012) and further studied in Yazdanshenasan et al. (2016).

The problem of joint percolation in both the primary and secondary networks was proposed and studied in Yemini et al. (2016a,b). Our main contribution in this paper is threefold. First, we introduce a re-parametrization of the problem, reducing the number of parameters from the five in Yemini et al. (2016a,b) to three. This enables us to summarize the behavior of the full model in a single plot, from which one can easily read off information about the original model. Second, we present simulation results on the critical radius for the existence of a left–right crossing, which approximates the critical radius for percolation; we also indicate the necessary steps towards a better approximation of the latter parameter. Finally, these simulation results, shown in Fig. 1, suggest several mathematical results on the dependence of the critical radius on the other

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http://dx.doi.org/10.1016/j.spl.2017.03.010
0167-7152/

parameters of the model. We state these results, together with sketches of some of their proofs, in the last main section. Rigorous proofs will appear in a forthcoming paper.

2. Mathematical background

We will use several facts and methods from continuum percolation. Most of these relate to the *Gilbert graph*, which was first defined and studied in Gilbert (1961), and we include a brief description of that model here. We also provide a short explanation of the methods that have been used to study the model, some of which were used in Yemini et al. (2016a,b), and some of which we will use ourselves. For more information, and rigorous proofs, the reader is encouraged to consult the books by Meester and Roy (1996), Penrose (2003), Bollobás and Riordan (2006), Haenggi (2012), and the survey article by Balister et al. (2009).

In Gilbert's model, we start with a Poisson process $\mathcal{P}$ of intensity one in the plane. These points form the vertices of an infinite graph G_r. The edges of G_r are obtained by joining two points of $\mathcal{P}$ if they lie at (Euclidean) distance less than r, where r is a fixed parameter.

The main quantity of interest for the Gilbert model is the *critical radius for percolation*. To define this, imagine fixing $\mathcal{P}$ and slowly increasing r, starting from $r = 0$. Initially, the graph G_r will consist of small components, whose vertices happen to lie close together, and isolated vertices. (Here, we use standard graph-theoretic terminology, so that a *component*, by definition, means a connected component.) As we increase r, these components will grow and merge, and at some point an infinite component I, containing a positive fraction $\theta(r) > 0$ of all the vertices in G_r, will appear. When this happens, we say that *percolation* occurs, or that the model G_r *percolates*. The fraction $\theta(r)$ of vertices in I can also be interpreted as the probability that a fixed vertex of G_r belongs to I, and, as r increases, $\theta(r)$ will naturally increase towards 1.

From a rigorous mathematical perspective, Kolmogorov's 0–1 law on tail events implies that, for any fixed value of r, the probability that G_r percolates (and also $\theta(r) > 0$) is either zero or one. In other words, if we consider several different instances of $\mathcal{P}$, and simultaneously increase r in each of them, at the same rate, then percolation occurs *at the same time* in each instance. Consequently, if we define r_{crit} as

$$r_{\text{crit}} = \sup\{r : \theta(r) = 0\},$$

then, for $r < r_{\text{crit}}$, G_r does not percolate (almost surely), and, for $r > r_{\text{crit}}$, G_r percolates (again, almost surely, i.e., with probability 1). As it happens, when $r = r_{\text{crit}}$, G_r does not percolate; this was established in Alexander (1996).

Given this, the next step is to obtain good bounds on r_{crit}. Currently the best known rigorous bounds, due to Hall (1985), are

$$0.833 < r_{\text{crit}} < 1.836.$$

These bounds are only slight improvements of Gilbert's original bounds from 1961, and were obtained using refinements of Gilbert's original methods. The lower bounds were obtained using branching processes, while the upper bounds come from comparison with a lattice percolation model, specifically, face percolation on a hexagonal lattice. More recently, Balister et al. (2005) used dependent percolation to show that, with confidence 99.99%,

$$1.1978 < r_{\text{crit}} < 1.1989.$$

(In detail, Balister et al. showed that, subject to a certain bound on a certain multidimensional integral, the stated bounds on r_{crit} hold; the integral itself was estimated using Monte Carlo methods, resulting in the stated confidence level.)

For more complicated models, such as the secrecy graph model (Sarkar and Haenggi, 2013), and the model considered in Yemini et al. (2016a,b), comparison with (dependent or independent) lattice percolation remains the main tool for bounding the various thresholds (indeed, it is used extensively in Yemini et al., 2016a,b). These comparisons work by superimposing an appropriately-sized lattice on the plane, and declaring a face F of the lattice "open" if F contains a point of $\mathcal{P}$. If the lattice spacing has been chosen correctly, then face percolation in the lattice implies percolation in the original model. Therefore, we can use classical bounds for lattice percolation thresholds to deduce that percolation occurs in the original model, for certain parameter values. The method can also be used to show that, for certain other parameter values, percolation does not occur; occasionally, one has to use dependent percolation to make the comparisons work, and this usually results in very weak bounds. Recent innovations include the *rolling ball method* of Balister and Bollobás (2016), and the *high confidence* method introduced in Balister et al. (2005), referred to above. Both these newer methods can also be adapted to other models; for instance they were used in Sarkar and Haenggi (2013) to study the secrecy graph.

The Gilbert model is primarily a model of a random geometric *graph*. However, there is a related *coverage process*, which we will make heavy use of in this paper. To define this coverage process $\mathcal{C}_r$, known as the *Gilbert disc model*, we start with a unit-intensity Poisson process $\mathcal{P}$ as before, but this time we place an open disc $B(p, r)$ of radius r around each point $p \in \mathcal{P}$. The connection between the Gilbert disc model and the Gilbert graph G_r is that graph-theoretic components in G_r correspond exactly to topological components of $\mathcal{C}_{r/2}$. If $\mathcal{C}_{r/2}$ has an infinite (topological) component, we extend our earlier terminology by saying that $\mathcal{C}_{r/2}$ percolates, which, by the above, occurs if and only if G_r percolates as well.

There are several quantities related to the Gilbert disc model which can be conveniently expressed in terms of the connection radius r. First, there is the *average coverage level* $\alpha_r = \pi r^2$, which represents both the average number of times a point of $\mathbb{R}^2$ is covered by $\mathcal{C}_r$ and also the average degree in G_r. Then there is the *reduced coverage level* $\alpha'_r = \alpha_{r/2} = \alpha_r/4$,