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Perturbation technique and method of fundamental solution to solve nonlinear Poisson problems

A. Tri [a], H. Zahrouni [b,*], M. Potier-Ferry [b]

[a] Laboratoire de Mécanique, Faculté des Sciences Aïn Chok, Université Hassan II Casablanca, Casablanca, Maroc

[b] Laboratoire d'Étude des Microstructures et de Mécanique des Matériaux, LEM3, CNRS, Université Paul Verlaine – Metz, Ile du Saulcy, 57045 Metz, France

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ABSTRACT

We show in this work that the Asymptotic Numerical Method (ANM) combined with the Method of Fundamental Solution (MFS) can be a robust algorithm to solve the nonlinear Poisson problem. The ANM transforms the nonlinear problem into a sequence of linear ones which can be solved by MFS. This last method consists of approximating the solution of the linear Poisson problem by a linear combination of fundamental solutions. Some examples are presented to show the efficiency of the proposed method.

1. Introduction

A few decades ago we have shown the efficiency of the ANM to compute the solution of nonlinear partial differential equations. Many applications have established the robustness of this method for nonlinear problems in solid and fluid mechanics, nonlinear vibrations, contact, large displacement and rotations, plasticity and other fields in physics [1–14].

ANM consists in computing the solution into power series with respect to a scalar parameter. This allows one to transform the nonlinear problem into a sequence of linear ones which have the same tangent operator. Consequently only one tangent matrix triangulation is needed to compute all the terms of the series. As the series have a limited convergence radius, the technique of Padé approximants is used to improve the validity range of the solution [7,8]. Up to now, ANM is generally associated to Finite Element Method to solve the resulting linear problems.

In recent years, there have been increasing interests in using meshfree techniques which aim to avoid the meshing restrictions encountered in the classical Finite Element Method. Several techniques have been proposed, but here, we are particularly interested in the so called Method of Fundamental Solutions (MFS) for the simplicity of its numerical implementation.

The main idea of this method consists of approximating the solution of the problem by a linear combination of fundamental solutions with respect to some source points which are located outside the domain. Then, the original problem is reduced to determining the unknown coefficients of the fundamental solutions by requiring the approximation to satisfy the boundary conditions.

MFS was first proposed by Kupradze and Aleksidze [15] and has been applied to many physical problems represented by linear differential equations, such as Laplace equation, Poisson's equation, eigenvalue problem, Helmholtz equation, Stokes equations, inverse problems, plate bending problems, etc. [16–26]. This method has been extended to solve some nonlinear problems [27-32]. It was mainly combined with iterative methods as Newton–Raphson method, Picard iteration [19,20] or concept of a matrix particular solution [22]. The association of MFS and ELM (Eulerian–Lagragian method) has been used to deal with nonlinear problems successfully, such as advection–diffusion equations [28], Burgers' equation and Navier–Stokes equations [29,30]. The Trefftz method and MFS have been intensively investigated by Balakrishnan and Ramachandran [22,23] for nonlinear problems in heat and mass transfer.

Many methods have been proposed in the last decade to improve MFS. In these papers, a key point is the introduction of various shape functions to discretize the considered linear problems. The Analog Equation Method (AEM) [21,31,40] allows solving linear equations even if a fundamental solution is not known, what is of high interest when dealing with nonlinear equations. The idea is to use shape functions that are solutions of another "analog" equation, but it is no longer a boundary-only technique as MFS. There are other interesting principles to create other shape functions, for instance by starting from Helmholtz operator instead of Laplacian or by finding a family of linear operators whose combination cancels the right hand side: this has led to the boundary knot method [34] and the boundary

* Corresponding author.
E-mail addresses: jalil@univ-metz.fr, hamid.zahrouni@univ-metz.fr (H. Zahrouni), michel.potier-ferry@univ-metz.fr (M. Potier-Ferry).

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particle method [37,38] that are in nature meshfree and boundary-only. It has also been proposed to define shape functions by Taylor series computed from the differential equation [31]. Likely the present ANM study could be extended by considering all these discretization techniques: indeed ANM is a generic procedure to transform a nonlinear problem into a family of linear ones and it works more or less independently of the discretization principle.

In the present work, the idea is to show that the association of ANM and MFS allows us to obtain an efficient algorithm to solve nonlinear Poisson problems. So we propose to solve the following problem

$$-\Delta u + u^3 = \lambda f \tag{1.1}$$

with the Dirichlet boundary conditions

$$u = \lambda g \tag{1.2}$$

where u is the unknown variable, λ a scalar parameter, f and g are given. The nonlinear terms of (1.1) can be chosen in more complicated forms; see the following contributions of ANM to solve problems involving strong nonlinearities [8–11].

The layout of this paper is as follows. In Section 2, we present the perturbation technique that transforms the nonlinear Eq. (1.1) into a sequence of linear Poisson problems. In Section 3, we show how to solve the resulting linear Poisson problems using MFS. In Section 4, we present some numerical results.

2. Perturbation technique

We aim to solve the nonlinear problems (1.1) and (1.2) by using the Asymptotic Numerical Method (ANM) which consists in associating a perturbation technique with a discretization method. The basic idea of ANM is first to set the problem to be solved into a quadratic form which is convenient to make easy the recurrence formulae, to expand the variables into power series and then to solve the resulting linear problems by a numerical procedure [2]. Assume that the nonlinear problem can be written in the following form:

$$R(U,\lambda) = L(U) + Q(U,U) - \lambda F = 0 \tag{2.1}$$

where R is the so-called residual vector, L and Q are a linear and a quadratic operators, F is a given vector and λ is a scalar parameter. The mixed vector U can hold several variables according to the considered problem. Next the variables (U,λ) are expanded into power series with respect to a path parameter "a":

$$U(a) - U_0 = \sum_{i=1}^{N} a^i U_i, \quad \lambda(a) - \lambda_0 = \sum_{i=1}^{N} a^i \lambda_i \tag{2.2}$$

where (U_0,λ_0) is a starting solution point and "N" is the truncation order which can be very large. Substituting (2.2) into (2.1) leads to a sequence of linear problems admitting the same tangent operator. One obtains for a given order "p"

$$L_t(U_p) = \lambda_p F + F_p^{nl} \tag{2.3}$$

where $L_t(.) = L(.) + 2Q(U_0,.)$is the tangent operator computed at the starting point (U_0,λ_0). At each order, the linear problem (2.3) is associated with a new right hand side term:

$$F_p^{nl} = -\sum_{i=1}^{p-1} Q(U_i, U_{p-i}) \tag{2.4}$$

These latter involve only a simple sum thanks to the quadratic framework of (2.1). Note that for order 1, the right hand side term $F_1^{nl}$is zero. To improve the validity range of the solution, the polynomial approximation (2.2) is replaced by Padé approximants [7,8]:

$$U(a) - U_0 = \sum_{i=1}^{N} Pad_i(a) a^i U_i, \quad \lambda(a) - \lambda_0 = \sum_{i=1}^{N} Pad_i(a) a^i \lambda_i \tag{2.5}$$

where $Pad_i(a)$ are rational fractions with the same denominator. Padé approximants are now commonly used because of their robustness and their low overhead in computing time.

It seems to be easy to adapt Eq. (1.1) to a quadratic form. One adds a new variable $\tilde{u} = u^2$ to set the problem into the following form:

$$\begin{cases} -\Delta u + u\tilde{u} = \lambda f & \text{in } \Omega \\ \tilde{u} = u^2 & \text{in } \Omega \\ u = \lambda g & \text{over } \partial\Omega \end{cases} \tag{2.6}$$

In this case, the vectors U, F and the operators L, Q are defined as

$$U = \begin{pmatrix} u \\ \tilde{u} \end{pmatrix}, \quad F = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad L = \begin{pmatrix} -\Delta u \\ \tilde{u} \end{pmatrix}, \quad Q(U,U) = \begin{pmatrix} u\tilde{u} \\ -u^2 \end{pmatrix} \tag{2.7}$$

Many choices are possible for the expansion parameter "a". We propose here to choose $a = \lambda - \lambda_0$ and to start from $(U_0,\lambda_0) = (0, 0)$. This leads to the following linear problems with $L_t = L = -\Delta$:

$$\text{For order 1}: \begin{cases} -\Delta u_1 = f & \text{in } \Omega \\ \tilde{u}_1 = 0 & \text{in } \Omega \\ u_1 = g & \text{over } \partial\Omega \end{cases} \tag{2.8}$$

$$\text{For order 2}: \begin{cases} -\Delta u_2 = 0 & \text{in } \Omega \\ \tilde{u}_2 = u_1^2 & \text{in } \Omega \\ u_2 = 0 & \text{over } \partial\Omega \end{cases} \tag{2.9}$$

$$\text{For order } k > 2: \begin{cases} -\Delta u_k = -\sum_{i=1}^{k-1} u_i \tilde{u}_{k-1} & \text{in } \Omega \\ \tilde{u}_k = \sum_{i=1}^{k-1} u_i u_{k-1} & \text{in } \Omega \\ u_k = 0 & \text{over } \partial\Omega \end{cases} \tag{2.10}$$

Note that for all these problems only the right hand side terms change. Classically, ANM associates the perturbation technique to Finite Element Method (FEM) to solve the resulting linear problems (2.10). Its effectiveness has been proven for many problems involving moderate or strong nonlinearities [6,9–13].

Many truncation orders can be chosen within ANM, depending on the physical problem, the number of degrees of freedom and the accuracy required by the user. Of course the domain of validity of the asymptotic solution is always smaller than the radius of convergence and it is generally rather large, say 60–80% of the radius of convergence for an order between 15 and 20. If Padé approximants are considered, the domain of validity is often larger and can be extended beyond the radius of convergence. For large scale problems involving thousands or millions of degrees of freedom, the truncation order can be chosen by considering the computational cost: an order of 15–20 is often optimal for small problems ($\sim 10^4$ dof), but larger orders (say 50) can be better for larger problems ($\sim 10^6$ dof) as illustrated in [41]. In the present paper, we only study very small problems and the computational cost is negligible. The truncation order will be re-discussed in Section 4, only with a view to maximize the domain of validity of the asymptotic solution. For more details about this discussion, refer to [3–5,8,41].

In the present paper, we show that ANM can be easily applied using MFS instead of FEM to solve the sequence of linear problems (2.10). The main idea of MFS is detailed in the next section.

3. Method of fundamental solution

In this section, we propose to solve by MFS the linear Poisson problems (2.8)–(2.10). Since the terms $\{\tilde{u}_i\}_{i=1,k}$ of the additional variable $\tilde{u}$ are known, these linear problems can be set into the