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A Gaussian expectation product inequality

Zhenxia Liu [a], Zhi Wang [b], Xiangfeng Yang [a,*]

[a] Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden
[b] School of Sciences, Ningbo University of Technology, Ningbo 315211, PR China

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ABSTRACT

Let $(X_1, \ldots, X_n)$ be any n-dimensional centered Gaussian random vector, in this note the following expectation product inequality is proved:

$$\mathbb{E}\prod_{j=1}^{n} f_j(X_j) \geq \prod_{j=1}^{n} \mathbb{E} f_j(X_j)$$

for functions f_j, $1 \leq j \leq n$, taking the forms $f_j(x) = \int_0^\infty \cos(xu)\mu_j(du)$, where μ_j, $1 \leq j \leq n$, are finite positive measures. The motivation of studying such an inequality comes from the Gaussian correlation conjecture (which was recently proved) and the Gaussian moment product conjecture (which is still unsolved). Several explicit examples of such functions f_j are given. The proof is built on characteristic functions of Gaussian random variables.

1. Introduction

Suppose that $(X_1, \ldots, X_n)$ is an n-dimensional centered Gaussian random vector. Gaussian expectation product inequality in the following form has been attracting considerable attention:

$$\mathbb{E}\prod_{j=1}^{n} f_j(X_j) \geq \prod_{j=1}^{n} \mathbb{E} f_j(X_j) \tag{1.1}$$

for suitable functions $f_j(x) \geq 0$. In this note, our aim is to study (1.1) under suitable restrictions on the functions $f_j(x)$, but not on the Gaussian random vector $(X_1, \ldots, X_n)$. For Gaussian expectation inequalities based on a Gaussian random vector with additional assumptions such as "associated" or "positively correlated", we refer to the recent Ref. Eisenbaum (2014).

Taking $f_j(x) = 1_{\{|x| \leq t_j\}}$, $t_j > 0$, the inequality (1.1) leads to

$$\mathbb{P}\left(|X_j| \leq t_j,\ 1 \leq j \leq n\right) \geq \prod_{j=1}^{n} \mathbb{P}\left(|X_j| \leq t_j\right),$$

which is a special case ($k = 1$) of the Gaussian correlation conjecture:

$$\mathbb{P}\left(|X_j| \leq t_j,\ 1 \leq j \leq n\right) \geq \mathbb{P}\left(|X_j| \leq t_j,\ 1 \leq j \leq k\right) \cdot \mathbb{P}\left(|X_j| \leq t_j,\ k+1 \leq j \leq n\right). \tag{1.2}$$

* Corresponding author.
E-mail addresses: zhenxia.liu@hotmail.com (Z. Liu), wangzhi1006@hotmail.com (Z. Wang), xiangfeng.yang@liu.se (X. Yang).

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This special case was obtained independently in Khatri (1967) and Šidák (1968). A full and detailed proof of the Gaussian correlation conjecture was recently given and verified in Royen (2014) and Latała and Matlak (2015). Taking $f_j(x) = |x|^{\alpha_j}, \alpha_j > 0$, the inequality (1.1) leads to the Gaussian moment product conjecture firstly formulated in Li and Wei (2012):

$$\mathbb{E}\prod_{j=1}^{n}|X_j|^{\alpha_j} \geq \prod_{j=1}^{n}\mathbb{E}|X_j|^{\alpha_j}. \tag{1.3}$$

So far this conjecture has been verified for only a few special cases such as $\alpha_j = 2$ for $j = 1, \ldots, n$; see Frenkel (2008). It was proved in Wei (2014) that the conjecture is true even for negative exponents $-1 < \alpha_j < 0$.

Now if we take $f_j(x) = \exp(-|x|^{p_j})$ for $p_j > 0$, then the inequality (1.1) involves an exponential term $\mathbb{E}\exp(-\sum_{j=1}^{n}|X_j|^{p_j})$ which is important in many studies. An exact equality for the special case $p_j = 2$ has been given as follows:

$$\mathbb{E}\exp\left(-\sum_{j=1}^{n}|X_j|^2\right) = (\det(I_n + 2C))^{-1/2},$$

where I_n is the $n \times n$ identity matrix, and $C = (c_{ij})_{1\leq i,j\leq n}$ is the covariance matrix of $(X_1, \ldots, X_n)$; see for instance Latała and Matlak (2015, Lemma 4) and Wei (2014, Theorem 3.1). Such an equality immediately implies that

$$\mathbb{E}\exp\left(-\sum_{j=1}^{n}|X_j|^2\right) \geq \prod_{j=1}^{n}\mathbb{E}\exp\left(-|X_j|^2\right)$$

using Hadamard's inequality (see Horn and Johnson, 1985). The case $p_j = 2$ is feasible since here we handle a quadratic form; the method is not applicable to general $p_j > 0$. Then a natural question is:

$$\mathbb{E}\exp\left(-\sum_{j=1}^{n}|X_j|^{p_j}\right) \geq \prod_{j=1}^{n}\mathbb{E}\exp\left(-|X_j|^{p_j}\right), \quad \text{for general } p_j > 0? \tag{1.4}$$

It is clear that the set $\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \exp\left(-\sum_{j=1}^{n}|x_j|^{p_j}\right) \geq r\right\}$ for $p_j \geq 1$ and any constant r is convex, therefore the Gaussian correlation inequality (1.2) (in an equivalent form; see Conjecture C' in Schechtman et al., 1998) immediately gives an affirmative answer to the question (1.4) when $p_j \geq 1$. For exponents $p_j < 1$, the question (1.4) remained unsolved. Motivated by this fact, in this note we prove the following expectation product inequality which can give an affirmative answer to the question (1.4) when $p_j < 1$.

Theorem 1.1. *For any n-dimensional centered Gaussian random vector* $(X_1, \ldots, X_n)$, *it is true that*

$$\mathbb{E}\prod_{j=1}^{n}f_j(X_j) \geq \prod_{j=1}^{n}\mathbb{E}f_j(X_j),$$

where $f_j(x) = \int_0^\infty \cos(xu)\mu_j(du)$, $1 \leq j \leq n$, *and* μ_j, $1 \leq j \leq n$, *are finite positive measures.*

An explicit family of functions f_j to which Theorem 1.1 can be applied is described as follows: $f_j(x)$, $1 \leq j \leq n$, are real-valued, positive definite, continuous at the origin and $f_j(0) = 1$ (where the reasoning is contained in the proof of part (ii) of Corollary 1.1). Another motivation of studying the inequality in Theorem 1.1 is from the Gaussian moment product conjecture (1.3). More precisely, let us consider the integral $\int_0^\infty u^{-\tau-1}(1-\cos(u))du$ for any $\tau \in (0, 2)$. This integral is finite and positive. Therefore, by setting $u = t|x|$,

$$|x|^\tau = c_\tau \int_0^\infty t^{-\tau-1}(1 - \cos(tx))\,dt$$

for some positive finite c_τ. Then Theorem 1.1 (with $f_1(x) = f_2(x) = \cos(x)$) implies

$$\begin{aligned}\mathbb{E}\left(|X_1|^{\alpha_1}|X_2|^{\alpha_2}\right) &= c_{\alpha_1}c_{\alpha_2}\int_0^\infty\int_0^\infty s^{-\alpha_1-1}t^{-\alpha_2-1}\mathbb{E}\left[(1-\cos(sX_1))(1-\cos(tX_2))\right]dsdt\\ &\geq c_{\alpha_1}c_{\alpha_2}\int_0^\infty\int_0^\infty s^{-\alpha_1-1}t^{-\alpha_2-1}\mathbb{E}\left(1-\cos(sX_1)\right)\mathbb{E}\left(1-\cos(tX_2)\right)dsdt\\ &= \mathbb{E}\left(|X_1|^{\alpha_1}\right)\mathbb{E}\left(|X_2|^{\alpha_2}\right).\end{aligned}$$

This is a special case of the Gaussian moment product conjecture (1.3) with $n = 2$ and $\alpha_1, \alpha_2 \in (0, 2)$. Although this special case can be obtained in many other elementary ways, the idea of using integral representations involving cosine functions leads to some hope of solving the Gaussian moment product conjecture (1.3) in this direction. The following corollary includes several more explicit functions f_j for which Theorem 1.1 can be applied.

Corollary 1.1. *For any n-dimensional centered Gaussian random vector* $(X_1, \ldots, X_n)$, *it is true that*