Statistics and Probability Letters 124 (2017) 64–68

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Are the Sweden Democrats really Sweden's largest party? A maximum likelihood ratio test on the simplex

J. Bergman *, B. Holmquist

Department of Statistics, Lund University, Box 743, SE-220 07 Lund, Sweden

ARTICLE INFO

Article history:
Received 10 June 2016
Received in revised form 8 January 2017
Accepted 9 January 2017
Available online 12 January 2017

MSC:
62F03
62H15
62P25

Keywords:
Isometric logratio transformation
Largest party
Maximum likelihood ratio test
Political polls
Polls, Sweden
Simplex

ABSTRACT

We introduce a maximum likelihood ratio test to test if a specific proportion is the greatest in a multinomial situation with a single measurement. The test is based on partitioning the parameter space and utilising logratio transformations.

1. Introduction

On 20 August 2015, the Swedish newspaper *Metro* ran the headline 'Now the Sweden Democrats are Sweden's largest party' (our translation) across its front page (Wallroth, 2015). From a journalistic point of view the headline was not surprising: 10 years ago the nationalistic party the Sweden Democrats (SD) had a voter share of 1%–2% and was hardly ever reported in the polls, and now there was a poll that gave the party the largest voter share of any party. A remarkable change indeed. However, from a statistical point of view the headline was intriguing: how do we test such a claim? Any introductory text book in statistics will tell you how to test if a proportion is greater than a specified value in a binomial situation. But in this case there is no specified value to test, and furthermore, Sweden has a multiparty system with, in practice, 8–10 competing parties, so this is a multinomial situation. So, how can we test if a specific share is greater than all the others?

One immediate approach would be to perform pairwise tests of the specific share against each of the others. However, to attain an overall level of significance, these tests need to be adjusted, e.g. with a Bonferroni correction. Apart from the general lack of elegance of such an approach, the procedure becomes less attractive when the number of parties increases; in a ten-party situation nine tests would be needed and for each test the significance level would have to be a mere 0.0055 for the overall significance level to be 0.05. A more serious objection is that such procedures do not incorporate the implicit

* Corresponding author.
E-mail addresses: jakob.bergman@stat.lu.se (J. Bergman), bjorn.holmquist@stat.lu.se (B. Holmquist).

http://dx.doi.org/10.1016/j.spl.2017.01.006

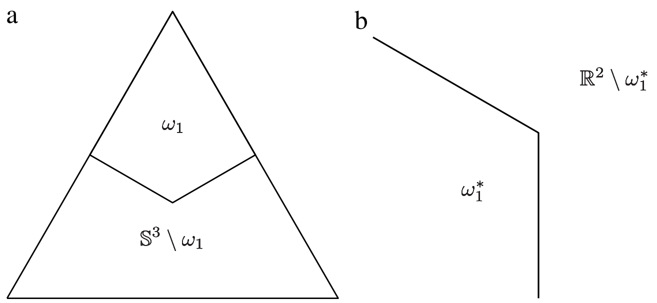

Fig. 1. The parameter space $\mathbb{S}^3$ is shown in (a) partitioned into the subspace ω_1, where p_1 is the largest part, and $\mathbb{S}^3 \setminus \omega_1$, where p_1 is not the largest part. The top vertex corresponds to $\boldsymbol{p} = (1, 0, 0)'$, the bottom left to $\boldsymbol{p} = (0, 1, 0)'$, and the bottom right to $\boldsymbol{p} = (0, 0, 1)'$. The boundary between ω_1 and $\mathbb{S}^3 \setminus \omega_1$, is the line from $\boldsymbol{p} = (1/2, 1/2, 0)'$, via $\boldsymbol{p} = (1/3, 1/3, 1/3)'$ to $\boldsymbol{p} = (1/2, 0, 1/2)'$. In (b) the corresponding parameter space in $\mathbb{R}^2$, partitioned into $\omega_1^* = \text{ILR}(\omega_1)$ and $\mathbb{R}^2 \setminus \omega_1^*$, is shown.

structure of the observed shares or frequencies; due to the fact that the shares need to sum to 1 or the frequencies to n, respectively, they are not independent but negatively correlated.

Instead of multiple tests, we would like one single test. We propose a maximum likelihood ratio test utilising the inherent properties of shares (proportions) to test the hypothesis. In Section 2 we introduce some notation, formalise the problem and discuss the properties of the parameter space, in Section 3 we derive the test and its properties. We apply the test to the newspaper article above in Section 4.

2. Voter shares and the simplex

Let $\boldsymbol{p} = [p_j]$ denote the vector of voter shares of the D parties in the electorate. (If there is a multitude of very small parties, the Dth share can represent the sum of all small parties.) Since $\boldsymbol{p}$ is non-negative and must sum to 1, the parameter space of $\boldsymbol{p}$ is the D-part simplex $\mathbb{S}^D$. Given a simple random sample of n respondents, the number of voters for each party $\boldsymbol{X}$ is a multinomial random variable with parameter $\boldsymbol{p}$. (Of course, $\boldsymbol{X}$ actually has a multivariate hypergeometric distribution, but we will assume that the population is large enough for the multinomial distribution to be an acceptable approximation.)

The statement that the ith share p_i is the greatest of the D shares is a relative statement, which, however, has absolute implications: a necessary condition is that $p_i > 1/D$ and a sufficient condition is that $p_i > 1/2$ (see Appendix A for proofs). We believe though that it is easier to consider the entire parameter space than to try to find explicit expressions for p_i. This means testing the hypotheses

$$\begin{aligned} H_0 &: \boldsymbol{p} \in \mathbb{S}^D \setminus \omega_i \\ H_1 &: \boldsymbol{p} \in \omega_i \end{aligned} \tag{1}$$

where ω_i is the subspace of $\mathbb{S}^D$ in which the ith part (share) is the greatest. The boundary between the two subspaces is the line, plane etc. where $p_i = p_j$ for at least one $j \neq i$ and all other parts are smaller. As an illustration, the parameter space $\mathbb{S}^3$ is depicted in Fig. 1(a) as a ternary diagram.

However, the simplex can pose practical problems due to the constraints on the parameters. Aitchison (1982) introduced the logratio transformations to resolve some of these issues. One popular choice of such transformation is the isometric logratio (ILR) transformation (Egozcue et al., 2003). It resolves the summation constraint of the simplex and transforms the problem to the real space $\mathbb{R}^{D-1}$. As an illustration, the subspaces in $\mathbb{R}^2$ corresponding to $\mathbb{S}^3 \setminus \omega_1$ and ω_1 are depicted in Fig. 1(b). There are many different conceivable ILR transformations, one example is the vector $\boldsymbol{y} = [y_j]$ where

$$y_j = \frac{1}{\sqrt{j(j+1)}} \log \frac{\prod_{k=1}^{j} p_k}{p_{j+1}^j}, \quad j = 1, \ldots, D-1. \tag{2}$$

3. A maximum likelihood ratio test

We propose that (1) is tested using a maximum likelihood ratio (MLR) test. (Here we follow the terminology used by e.g Garthwaite et al. (2002, Sec. 4.6).) This means finding the maximum value of the likelihood in the restricted parameter space under H_0 and comparing this with the maximum value if the parameter space is not restricted. As the sample consists of only one observation, the likelihood function equals the probability function

$$L(\boldsymbol{p}|\boldsymbol{x}) = \frac{n!}{x_1! \cdots x_D!} p_1^{x_1} \cdots p_D^{x_D}. \tag{3}$$