Statistics and Probability Letters 124 (2017) 97–100

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Decomposing aggregate risk into marginal risks under partial information: A top-down method

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ARTICLE INFO

Article history:
Received 26 October 2016
Received in revised form 9 January 2017
Accepted 17 January 2017
Available online 23 January 2017

Keywords:
Aggregate risk model
Covariance matrix
Partial information
Top-down

ABSTRACT

This paper proposes a method to decompose a square-integrable random variable into any number of marginal random variables under partial information restrictions. An executable algorithm and a concrete example of the capital allocation problem are also provided.

1. Introduction

Stochastic modeling has become more and more popular in risk management since the groundbreaking work of Samuelson (1965) and Black and Scholes (1973). In the practice of quantitative risk management, modeling the aggregate risk model $S = X_1 + X_2 + \cdots + X_n$ is a fundamental problem, where S is the aggregate risk and $X_1, X_2, \ldots, X_n$ are the marginal risks. There are generally two directions to deal with this problem, one is "bottom-up" (see Duffie and Garleanu, 2001 for the intuitive insight on "bottom-up") and the other is "top-down" (see Giesecke et al., 2011 for "top-down" method).

From the angle of the "bottom-up", the problem consists of two much more fundamental aspects, one is the model for the marginal risks and other is the model for the dependence structure among them. For the model for marginal risks, practical users usually adopt the parameter models such as Gaussian, Beta and Gamma distributions, see Shao (2003) for references. For the dependence structure, practical users always describe the dependence structure by a copula function, which is a n-dimensional distribution with [0, 1] uniform marginal distributions, see Nelsen (2006) and McNeil et al. (2015) for more introductions. The most commonly used copula functions include the Gaussian copula (Li, 2000 applied it to the pricing of CDO), student-t copula and Archimedean copula. Please note that once marginal risks are given, the dependence structure among them is then of critical importance for the aggregate risk S. Wang and Wang (2011) and Wang et al. (2013) discussed the worst scenarios of the Value-at-Risk of the aggregate risk under dependence uncertainty but with known marginal distributions. Based on the information of the marginals and the dependence, the aggregate risk S becomes calculable, either by simulation or explicit formulation. However, there are often no closed-form expressions for distributions of the aggregate risks since it is always involved in a multiple integral,[1] sometimes the simulation can even be relatively difficult. Given these considerations, proposing a brief approach to substitute the "bottom-up" method is of crucial necessity from both viewpoints of industry practice and academic research.

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[1] For example when $n = 2$, $\mathbb{P}(X_1 + X_2 \le t) = \int_{-\infty}^{+\infty} \frac{\partial C}{\partial v}(F_{X_1}(t - x), F_{X_2}(x))\, dF_{X_2}(x)$ if the copula $C(u, v)$ of X_1 and X_2 is absolutely continuous.

http://dx.doi.org/10.1016/j.spl.2017.01.015
0167-7152/

In this paper, a "top-down" method that enables us to decompose the aggregate risk into marginal risks under partial information restrictions is proposed. Different from the "bottom-up" method, the "top-down" method will first model the aggregate risk and then thin it into a sum of marginal risks under partial information restrictions, namely, it first assumes the aggregate risk S and then assumes expectations and the covariance matrix of the marginal risks, see He et al. (2010), Kaas and Goovaerts (1986) and Popescu (2005) for more literature on partial information assumptions. With these assumptions, the "top-down" method enables us to find n (any positive integer) random variables $X_1, X_2, \ldots, X_n$ with the given expectations and covariance matrix such that $S = X_1 + X_2 + \cdots + X_n$, what is more important, the marginal risks $\{X_i\}_i$ all have closed-form solutions by the "top-down" method and it is exactly the closed-form solution that makes the simulation of the aggregate risk model much easier compared with the "bottom-up" method, especially when the number n of the marginal risks is very large.

The "bottom-up" method provides us an accurate description of the aggregate risk model, however, the "top-down" method is actually an approximation method of the much more elaborate "bottom-up" method, by substituting the distributions of marginal risks and the dependence structure with the mean and the covariance respectively.

The rest of this paper is structured as follows: Section 2 provides the main decomposition method, Section 3 gives an executable algorithm and applies it to a concrete example, Section 4 concludes the whole paper.

2. Decomposition method

All the random variables in this paper are defined in the same non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Now let us consider the standard aggregate risk model:

$$S = X_1 + X_2 + \cdots + X_n.$$

In this section, the viewpoint of the "top-down" is adopted: first assume that the aggregate risk S is known and then decompose S into a sum of marginal risks by restrictions of expectations and covariance of marginal risks.

Lemma 2.1. *Let S be a square-integrable random variable defined on a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then for any $n \in \mathbb{N}_+$ there exist random variables $\{\delta_i\}_{i=1}^{n-1}$ defined on the same probability space such that $S, \delta_1, \delta_2, \ldots, \delta_{n-1}$ are pairwise non-correlated.*

Proof. Without losing generality, we assume that $\mathbb{E}[S] = 0$. Let $\mathcal{D} = \{X : \mathbb{E}[X] = 0,\ \mathbb{E}[X^2] < +\infty\}$ and define an equivalence relation in $\mathcal{D}$: $X \sim Y \Leftrightarrow X = Y$ *a.s.* It is easy to see that $\mathcal{D}_0 = \mathcal{D}/\sim$ is a vector space on domain $\mathbb{R}$. Then define an inner product operation on $\mathcal{D}_0 \times \mathcal{D}_0$: $\langle X, Y\rangle := \mathrm{COV}(X, Y) = \mathbb{E}[XY]$, obviously, $\mathcal{D}_0$ is an inner product space under the operator $\langle X, Y\rangle$. Therefore, by Gram–Schmidt Orthogonalization, it is easy to expand $\{S\}$ to an orthogonal family $\{S, \delta_1, \delta_2, \ldots, \delta_{n-1}\} \subseteq \mathcal{D}_0$ since the dimension of $\mathcal{D}_0$ on $\mathbb{R}$ is infinite. Hence, $\{S, \delta_1, \delta_2, \ldots, \delta_{n-1}\}$ is a pairwise non-correlated family. □

Theorem 2.1. *Given any square-integrable random variable S on a non-atomic space and $n \in \mathbb{N}_+$, for any n expectations $\{\mu_i\}_{i=1}^n$ with $\sum_i \mu_i = \mathbb{E}[S]$ and any $n \times n$ semi-positive definite symmetrical matrix $\Sigma = (\sigma_{ij})_{ij}$ with $\sum_{i,j} \sigma_{ij} = \mathrm{Var}[S]$, there exists a random vector $\vec{X} = (X_1, X_2, \ldots, X_n)$ such that:*

$$S = X_1 + X_2 + \cdots + X_n,$$

in which $\mathbb{E}[X_i] = \mu_i$ for $i = 1, 2, \ldots, n$ and $\mathbb{E}[(\vec{X} - \mathbb{E}[\vec{X}])^\top(\vec{X} - \mathbb{E}[\vec{X}])] = \Sigma$.

Proof. Without losing generality, we assume that $\mathbb{E}[S] = 0$, $\mathrm{Var}[S] = 1$ and $\mu_i = 0$ for $i = 1, 2, \ldots, n$. Lemma 2.1 allows us to find $n - 1$ random variables $\delta_1, \delta_2, \ldots, \delta_{n-1}$ such that $S, \delta_1, \delta_2, \ldots, \delta_{n-1}$ are non-correlated, with normalization we can assume that $\mathbb{E}[\delta_i] = 0$ and $\mathrm{Var}[\delta_i] = 1$ for $i = 1, 2, \ldots, n-1$.

Now we will use $S, \delta_1, \delta_2, \ldots, \delta_{n-1}$ to construct X_i for $i = 1, 2, \ldots, n$, in other words, we will express X_i as a linear combination of this non-correlated family. Then the proof turns to find a $n \times n$ matrix $A = (a_{ij})_{ij}$ with the following three Restrictions 1–3 such that:

$$A(S, \delta_1, \delta_2, \ldots, \delta_{n-1})^\top = (X_1, X_2, \ldots, X_n)^\top. \tag{1}$$

Restriction 1. $\mathbb{E}[X_i] = 0 \iff a_{i1}\mathbb{E}[S] + \sum_{j=2}^{n-1} a_{ij}\mathbb{E}[\delta_{j-1}] = 0$ for $i = 1, 2, \ldots, n$.

Restriction 2. $\mathbb{E}[(\vec{X} - \mathbb{E}[\vec{X}])^\top(\vec{X} - \mathbb{E}[\vec{X}])] = \Sigma \iff AA^\top = \Sigma$.

Restriction 3. $S = X_1 + X_2 + \cdots + X_n \iff (1, 1, \ldots, 1)A = (1, 0, 0, \ldots, 0)$.