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Stochastic Processes and their Applications 127 (2017) 3661–3688

stochastic processes and their applications

www.elsevier.com/locate/spa

A stroll along the gamma

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Received 7 November 2016; accepted 13 March 2017
Available online 22 March 2017

Abstract

We provide the first in-depth study of the Laguerre interpolation scheme between an arbitrary probability measure and the gamma distribution. We propose new explicit representations for the Laguerre semigroup as well as a new intertwining relation. We use these results to prove a local De Bruijn identity which hold under minimal conditions. We obtain a new proof of the logarithmic Sobolev inequality for the gamma law with $\alpha \geq 1/2$ as well as a new type of HSI inequality linking relative entropy, Stein discrepancy and standardized Fisher information for the gamma law with $\alpha \geq 1/2$.

MSC: 60E15; 26D10; 60B10

Keywords: De Bruijn identity; Entropy; Fisher information; Gamma approximation; Semigroup interpolation; Smart path; Representation formulae

1. Introduction

The Laguerre process is a prototypical example of a Markov diffusion process whose fundamental properties are well-understood. One can define it as the unique strong solution of the following stochastic differential equation:

$$\forall t \geq 0, \quad dX_t^x = \sqrt{2X_t^x}dB_t + (\alpha - \lambda X_t^x)dt,$$

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http://dx.doi.org/10.1016/j.spa.2017.03.012

where $X_0^x = x \in (0, +\infty)$, $\{B_t\}$ is a standard Brownian motion and $\alpha, \lambda > 0$. As the Ornstein–Uhlenbeck and the Jacobi processes, it is naturally associated with a family of orthogonal polynomials, namely the Laguerre polynomials. Actually since the work of O. Mazet [22], it is known that these three continuous stochastic processes are essentially the only symmetric diffusions (in the sense of Definition 1.11.1 of [4]) on the real line for which the generator is diagonalizable with respect to orthogonal polynomials. As such, their full study is of importance. However, the Ornstein–Uhlenbeck process stands somewhat apart from the Laguerre and the Jacobi ones. Indeed, the Ornstein–Uhlenbeck semigroup enjoys a particularly useful representation, namely the Mehler integral representation (see formula 2.7.3 of [4]). This integral representation is of central importance for problems where a Gaussian interpolation scheme argument can be used and several identities for the Ornstein–Uhlenbeck process can be deduced from it. For example, the proof of the Gaussian logarithmic Sobolev inequality of L. Gross [18] reduces to a simple Cauchy–Schwarz argument thanks to this integral representation. It has been used recently as well in refinement of logarithmic Sobolev inequalities in the context of Stein's method (see [25,20] and below) and in the quantification of estimates for the deficit of the Gaussian logarithmic Sobolev inequality (see [8,15]). This Gaussian interpolation scheme argument is as well pivotal for the proof of entropic central limit theorem, as in [1,5–7,10,19,28]. In the Laguerre and Jacobi cases, they lack such integral representations of the actions of corresponding semigroups. In this paper, we provide a full study of the Laguerre process through new stochastic representations of the interpolation scheme along the Laguerre dynamic.

Before moving to a presentation of the results, we introduce some notations. We denote by $\gamma_{\alpha,\lambda}(.)$ the density of the gamma law with parameters (α, λ) given by:

$$\forall u > 0, \quad \gamma_{\alpha,\lambda}(u) = \frac{\lambda^\alpha}{\Gamma(\alpha)} u^{\alpha-1} \exp(-\lambda u).$$

Let $P_t^{(\alpha,\lambda)}$, $\mathcal{L}_{\alpha,\lambda}$ and $\Gamma_{\alpha,\lambda}$ be the Laguerre semigroup, the Laguerre generator and the carré du champs operator naturally associated with the gamma probability measure (which is invariant and reversible under the Laguerre dynamic). These operators are given respectively by the following formulae on subsets of $L^2\big(\mathbb{R}_+^*, \gamma_{\alpha,\lambda}(u)du\big)$ (see e.g. [17]):

$$\begin{aligned}
P_t^{(\alpha,\lambda)}(f)(x) &= \frac{\lambda^{1-\alpha}\Gamma(\alpha)e^{\lambda t}}{e^{\lambda t}-1} \int_0^{+\infty} f(u) \left(\frac{e^{\lambda t}}{xu}\right)^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda}{e^{\lambda t}-1}(x+u)\right) \\
&\quad \times I_{\alpha-1}\left(\frac{2\lambda\sqrt{uxe^{\lambda t}}}{e^{\lambda t}-1}\right)\gamma_{\alpha,\lambda}(du), \\
\mathcal{L}_{\alpha,\lambda}(f)(u) &= u\frac{d^2 f}{du^2}(u) + (\alpha - \lambda u)\frac{df}{du}(u), \\
\Gamma_{\alpha,\lambda}(f)(u) &= u\left(\frac{df}{du}(u)\right)^2,
\end{aligned}$$

where $I_{\alpha-1}(.)$ is the modified Bessel function of the first kind of order $\alpha - 1$ (see [27] Formula 10.25.2 for a definition).

We introduce as well the Fisher information structure naturally induced by the geometry of this Markov diffusion process. For any positive random variable Y with smooth enough density f_Y and score function $\rho_Y(u) = \partial_u\big(\log(f_Y(u))\big)$, the relative Fisher information of Y with respect