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Continuous spin models on annealed generalized random graphs

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Abstract

We study Gibbs distributions of spins taking values in a general compact Polish space, interacting via a pair potential along the edges of a generalized random graph with a given asymptotic weight distribution P, obtained by annealing over the random graph distribution.

First we prove a variational formula for the corresponding annealed pressure and provide criteria for absence of phase transitions in the general case.

We furthermore study classes of models with second order phase transitions which include rotation-invariant models on spheres and models on intervals, and classify their critical exponents. We find critical exponents which are modified relative to the corresponding mean-field values when P becomes too heavy-tailed, in which case they move continuously with the tail-exponent of P. For large classes of models they are the same as for the Ising model treated in Dommers et al. (2016). On the other hand, we provide conditions under which the model is in a different universality class, and construct an explicit example of such a model on the interval.

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1. Introduction

Spin models on random graphs are interesting for many reasons. Random graphs can for example serve as models for complex networks such as social, information, technological and biological networks [37]. Spins can e.g. describe an opinion or an internal state of a person [33] or neurons in the brain [19]. The interaction between the behavior of spin models and the properties of the random graph is of special interest, see for example [15] for an overview of (often non-rigorous) results in the physics literature.

Also in the mathematics community there has been a large interest in such models recently. Especially the ferromagnetic Ising model on random graphs has attracted a lot of attention, both in equilibrium [6,11–13,20,21] and out of equilibrium, i.e., the dynamics of this model [10,14,35]. In this model, spins can only take two values and the spins tend to align. A spin glass model on the Erdős–Rényi random graph was studied in [23]. In such models the couplings between two vertices are chosen w.r.t. a symmetric distribution independently over the pairs, so that both ferromagnetic and antiferromagnetic couplings are present. This makes them harder to study.

When spins can take more than two values, the models often become more difficult to analyze, see, e.g., [4,5,7,8] for some results where spins can take a finite number of values.

In this paper, we study a much more general setting where spins can take values in a general compact Polish space, in particular they can be continuous. Also the pair interaction potential between two neighboring spins is only assumed to be bounded. We study such models on *generalized random graphs* in the *annealed* setting.

In *generalized random graphs*, each edge (or bond) is present independently with probability which is (essentially) proportional to the product $w_i w_j$ where $w_i > 0$ is a quenched weight variable associated to the site i. Hence w_i can be seen as the affinity of the site i to form connections. See [3] for an extensive analysis of this model. We thereby assume that the empirical weight distribution at finite N converges in distribution to the distribution P of a limiting variable W taking values in the positive reals, such that the second moment converges, too.

By *annealing* we average the exponential of the Hamiltonian of the spin model under investigation over the random graph. Using the language of social networks, working in this annealed framework means that we are considering an equilibrium distribution in a regime where making (or losing) friends is happening on a faster time-scale than opinion-forming. There is also an interest in annealed spin models on random graphs in the context of quantum gravity, see e.g. [36].

A similar set-up has been studied for the particular case of the Ising model in [11,21]. In [21], the pressure is computed, showing that it is a function of the solution to some fixed point equation. Also a central limit theorem for the total spin is given. In [11], it was found that depending on tail-behavior of the limiting weight distribution P the transition stays second order, but the critical exponents change from their mean-field values when the weights become too heavy-tailed to values which are computable in terms of the tail-exponent. This shows that the annealed Ising model on random graphs is in the same universality class as its quenched counterpart [13].

In the present work, we prove that the pressure also exists in our setting with more general state spaces and interactions and show that it satisfies a variational principle. We also investigate the critical behavior of a class of models with second order phase transition (at which order appears continuously). These models are rotator models which have $O(q)$-symmetry and models on the