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On an approach to boundary crossing by stochastic processes

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Abstract

In this paper we provide an overview as well as new (definitive) results of an approach to boundary crossing. The first published results in this direction appeared in de la Peña and Giné (1999) book on decoupling. They include order of magnitude bounds for the first hitting time of the norm of continuous Banach-Space valued processes with independent increments. One of our main results is a sharp lower bound for the first hitting time of càdlàg real-valued processes $X(t)$, where $X(0) = 0$ with arbitrary dependence structure: $ET_r^\gamma \geq \int_0^1 \{a^{-1}(r\alpha)\}^\gamma d\alpha$, where $T_r = \inf\{t > 0 : X(t) \geq r\}, a(t) = E\{\sup_{0\leq s\leq t} X(s)\}$ and $\gamma > 0$. Under certain extra conditions, we also obtain an upper bound for ET_r^γ. As the main text suggests, although T_r is defined as the hitting time of $X(t)$ hitting a level boundary, the bounds developed can be extended to more general processes and boundaries. We shall illustrate applications of the bounds derived for additive processes, Gaussian Processes, Bessel Processes, Bessel bridges among others. By considering the non-random function $a(t)$, we can show that in various situations, $ET_r \approx a^{-1}(r)$.

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1. Introduction

One of the problems of wide interest in the study of stochastic processes involves estimation of $E[T_r]$, the expected time at which a process crosses a boundary r. Several tools have been developed for this purpose including Wald's equations [14], Doob's optional sampling theorem and Burkholder–Davis–Gundy inequality [3]. Our current discussion follows the setting discussed in de la Peña and Giné [7]. To motivate our development, we first present a simplified example where the underlying process is non-random, as is $f(t) \triangleq \sup_{0\le s\le t} x(s)$. The time of interest t_r equals the first time $t \ge 0$ such that $x(t)$, hence, $f(t)$ reaches or exceeds a fixed horizontal level $r > 0$.

Then, it is easy to see that, in the case of continuous functions,

$$f(t_r) = r \quad \text{and} \quad t_r = f^{-1}(r).$$

An immediate question is how to relate the expected hitting time when X_t is a random stochastic process.

To facilitate the discussion, let $\{X(t)\}_{t\ge 0}$, $X(0) = 0$, be a càdlàg process with first passage time across a level boundary $T_r = \inf\{t > 0 : X(t) \ge r\}$, and $T_r = \infty$ when $X(t) < r$ for all $t, r > 0$. In general, there is no explicit closed form solution for $E[T_r]$ due to the complicated model/dependence structure amongst observations. Let $a(t) = EM(t)$ where $M(t) = \sup_{0\le s\le t} X(s)$. We are interested in obtaining bounds for $E[T_r]$ as functions of $a(t)$. [6] first investigated this problem. Our results extend beyond stopping times for level boundaries. That is because we may embed the boundaries in a transformed process where level boundaries again apply (see Remark 2.2).

To the best of our knowledge, typical methods of obtaining densities or moments of the hitting time assume full knowledge of the distribution or very specific form of bounds to be hit, most of which are level boundaries. In contrast, our approach is based on an approximate knowledge of moments of the maximal process, which are either available in many popular processes for modeling or can be estimated through the empirical observations. In this paper, we obtain a sharp universal lower bound for $E\{g(T_r)\}$ for any non-decreasing function g. The main idea inherits the spirit of a natural extension of the concept of boundary crossing by non-random functions to the case of random processes. The maximal process $a(t)$ can be intuitively interpreted as a natural clock for all processes with the same $a(t)$.

The rest of the paper is organized as follows: In Section 2, we obtain our lower bound on $E[g(T_r)]$. Section 3 elaborates some possible extensions of our methodology that can handle situations in which the expected first hitting time is hard to obtain. Section 4 concludes the paper.

2. Main results

For the time being, we assume that $a(t)$ is continuous and strictly increasing so that $a^{-1}(\cdot)$ is unambiguously defined. We shall later relax this restriction and show how to extend the result to all càdlàg processes $X(t)$ with $X(0) = 0$. $a^{-1}(s)$ is unambiguously defined for $0 < s \le r$.

Theorem 2.1. (i) *Let $X(t)$ be a càdlàg stochastic process with $X(0) = 0$. Denote $a(t) = E\{\sup_{0\le s\le t} X(s)\}$ and $T_r = \inf\{t \ge 0 : X(t) \ge r\}$, where $r > 0$. For g non-decreasing,*

$$Eg(T_r) \ge \int_0^1 g\left\{a^{-1}(r\alpha)\right\} d\alpha. \tag{1}$$