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A law of the iterated logarithm for Grenander's estimator

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Abstract

In this note we prove the following law of the iterated logarithm for the Grenander estimator of a monotone decreasing density: If $f(t_0) > 0$, $f'(t_0) < 0$, and f' is continuous in a neighborhood of t_0, then

$$\limsup_{n\to\infty} \left(\frac{n}{2\log\log n}\right)^{1/3} (\widehat{f}_n(t_0) - f(t_0)) = \left|f(t_0)f'(t_0)/2\right|^{1/3} 2M$$

almost surely where

$$M \equiv \sup_{g\in\mathcal{G}} T_g = (3/4)^{1/3} \quad \text{and} \quad T_g \equiv \operatorname*{argmax}_u \{g(u) - u^2\};$$

here $\mathcal{G}$ is the two-sided Strassen limit set on $\mathbb{R}$. The proof relies on laws of the iterated logarithm for local empirical processes, Groeneboom's switching relation, and properties of Strassen's limit set analogous to distributional properties of Brownian motion; see Strassen [26].

MSC: 60F15; 60F17; 62E20; 62F12; 62G20

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1. Introduction: the MLE of a monotone density

Nonparametric estimation of a monotone density was first considered by Grenander [9,10]. Suppose that $X_1, \ldots, X_n$ are i.i.d. with distribution function F on $[0, \infty)$ having a decreasing density f. Grenander showed that the maximum likelihood estimator $\hat{f}_n$ of f is the (left-) derivative of the least concave majorant of the empirical distribution function $\mathbb{F}_n$ (see Figs. 1 and 2)

$$\widehat{f_n} = \{\text{left derivative of the least concave majorant of } \mathbb{F}_n\}.$$

The asymptotic distribution of $\hat{f}_n(t_0)$ at a fixed point t_0 with $f'(t_0) < 0$ was obtained by Prakasa Rao [22,23], and given a somewhat different proof by Groeneboom [11]; also see [13, sections 3.2 and 3.6]. If $f'(t_0) < 0$ and f' is continuous in a neighborhood of t_0, then

$$n^{1/3}(\widehat{f_n}(t_0) - f(t_0)) \to_d \left|\frac{1}{2} f(t_0) f'(t_0)\right|^{1/3} 2\mathbb{Z}, \tag{1}$$

where

$$\begin{aligned} 2\mathbb{Z} &= \text{slope at 0 of the least concave majorant of } W(t) - t^2 \\ &\stackrel{d}{=} \text{slope at 0 of the greatest convex minorant of } W(t) + t^2 \\ &\stackrel{d}{=} 2 \operatorname*{argmin}_{t\in\mathbb{R}}\{W(t) + t^2\}; \end{aligned} \tag{2}$$

here $\{W(t) : t \in \mathbb{R}\}$ is a two-sided Brownian motion process starting at 0. In fact, the convergence in (1) can be extended to weak convergence of the (local) Grenander process as follows. Let $\{\mathbb{S}_{a,b}(t) : t \in \mathbb{R}\}$ denote the slope process corresponding to the least concave majorant of $X_{a,b}(t) = aW(t) - bt^2$, with $a = \sqrt{f(t_0)}$ and $b = |f'(t_0)|/2$. Then for fixed t_0 with $f'(t_0) < 0$ and f' continuous in a neighborhood of t_0,

$$n^{1/3}(\hat{f}_n(t_0 + n^{-1/3}t) - f(t_0)) \Rightarrow \mathbb{S}_{a,b}(t)$$

in the Skorokhod topology on $D[-K, K]$ for every finite $K > 0$; see e.g. [12,18], and [17]. Groeneboom [12] gives a complete analytic characterization of the limiting distribution $\mathbb{Z}$ and further, the distributional structure of the process $\mathbb{S}$. The distribution of $\mathbb{Z} = \mathbb{S}(0)/2$ has been studied numerically by Groeneboom and Wellner [16] which relies heavily on [11,12]. Balabdaoui and Wellner [4] show that the distribution of $\mathbb{Z}$ is log-concave. Note that there is an "invariance principle" involved here: the centered slope of the least concave majorant of $\mathbb{F}_n$ converges weakly to a constant times the slope of the least concave majorant of $X(t) = W(t) - t^2$. We can regard the slope in this Gaussian limit problem, $2\mathbb{Z}$, as an "estimator" of the slope of the line $2t$ in the Gaussian problem of "estimating" the "canonical" linear function $2t$ in "Gaussian white noise" $dW(t)$ since

$$dX(t) = 2t dt + dW(t).$$

2. A law of the iterated logarithm for the Grenander estimator

Our main goal is to prove the following Law of the Iterated Logarithm (LIL) for the Grenander estimator corresponding to the limiting distribution result in (1).